Black hole scattering: the self-force perspective

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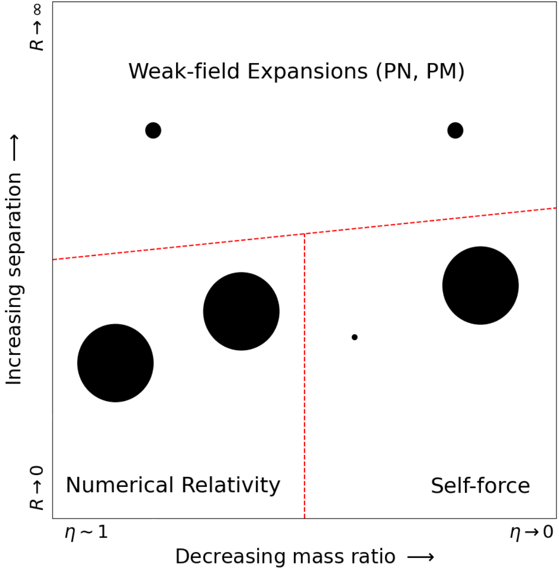
General-relativistic 2-body problem: modelling approaches



Gravitational self-force framework

Metric of the physical spacetime is expanded about a background spacetime as a series in $q := m/M \ll 1$,

$$g^{\rm phys}_{\alpha\beta} = g_{\alpha\beta} + q h^{(1)}_{\alpha\beta} + q^2 h^{(2)}_{\alpha\beta} + \ldots$$

- 0SF: Background metric $g_{\alpha\beta}$. Smaller object moves along fixed background geodesic.
- 1SF: Perturbation $h^{(1)}_{\alpha\beta}$ sourced by point particle on fixed background geodesic. Leading order self-force $\propto q$.
- 2SF: Perturbation $h^{(2)}_{\alpha\beta}$ sourced by particle on 1SF-perturbed trajectory. Gives rise to additional self-force terms $\propto q^2$.

Particle description derived, not assumed.

1SF equation of motion

- Metric perturbation may be split into regular and singular fields,
  [Detweiler & Whiting 2003]

$$h_{\alpha\beta} = h^{R} + h^{S},$$

defined in terms of certain acausal Green's functions.

- Only $h^{R}_{\alpha\beta}$ contributes to the self-force. For example, at 1SF order,

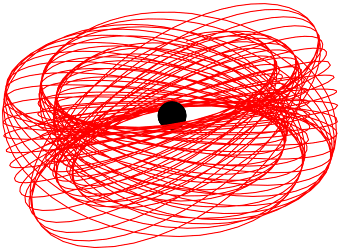
$$\frac{Du^{\alpha}}{d\tau} = q\nabla^{\alpha\beta\gamma} h^{R(1)}_{\beta\gamma}\Big|_{z(\tau)} + O(q^2) := qF^{\alpha},$$

where

$$\nabla^{\alpha\beta\gamma} h_{\gamma\beta} := -\frac{1}{2}\left(g^{\alpha\beta} + u^{\alpha}u^{\beta}\right) u^{\gamma}u^{\delta}\left(2\nabla_{\delta}h_{\beta\gamma} - \nabla_{\beta}h_{\gamma\delta}\right).$$

Extreme mass ratio inspirals (EMRIs)

Created using KerrGeodesics package from BHP toolkit.



- Highly asymmetric compact binaries. Typical mass ratios

$$q \sim \frac{10 M_\odot}{10^6 M_\odot} = 10^{-5} \ll 1$$

- Inspiral slow compared to orbital periods:

$$T_{\mathrm{RR}} \sim T_{\mathrm{orb}}/q \gg T_{\mathrm{orb}}.$$

- Large number of gravitational wavecycles in LISA band before merger:

$$N_{\mathrm{orb}} \sim 1/q \sim 10^5.$$

- Orbital dynamics complicated. Geodesics tri-periodic and generically ergodic.
- EMRIs offer a precision probe of strong-field geometry around black-holes.

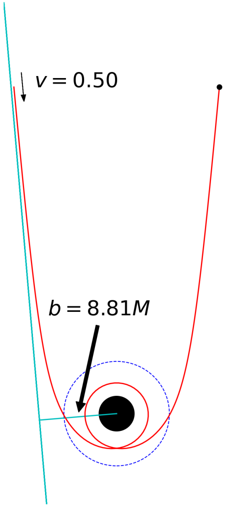
Hyperbolic scattering

- Scattering is a good probe of the 2-body dynamics: probe the sub-ISCO region at arbitrarily low energy with minimal parameter tuning.
- Motion begins and ends in the strong-field: gauge invariant observables such as the *scatter angle*,

$$\chi := \varphi_{\rm out} - \varphi_{\rm in} - \pi.$$

- *Universality:* can deduce information about (more astrophysically relevant) bound orbits from scatter data. E.g.

$$\chi(E, L) \mapsto H_{\rm EOB}(E, L)$$



[Damour 1609.00354, 1710.10599]

Post-Minkowskian scattering

- Scatter angle expanded in powers of GM/b,

$$\chi = \sum_{n=0}^{\infty} \chi_{\mathrm{nPM}} \left(\frac{GM}{b}\right)^n .$$

- Rapid progress in computing the PM expansion in recent years, driven by the adoption of techniques from high energy physics
- State of the art results at 5PM order using worldline QFT. [Driesse et al 2403.07781, Driesse et al 2601.16256]
- Boundary-to-bound relations map scatter observables to bound observables. [Kälin and Porto 1910.03008; Cho, Kälin and Porto 2112.03976, ...]

Scattering in the SF limit

- Scatter angle expanded in powers of the mass ratio,

$$\chi(v, b) = \chi^{0\mathrm{SF}}(v, b) + q\chi^{1\mathrm{SF}}(v, b) + O(q^2).$$

  Split between geodesic term $\chi^{0\mathrm{SF}}$ and self-force *correction* $\chi^{1\mathrm{SF}}$ defined at fixed (v, b).

- Complementary domain of validity: small mass ratio but arbitrary velocity
  - Benchmark (or *resum*) weak-field PM results in the strong-field.

- Self-force can be used to determine high-order PM terms: [Damour 1912.02139]

$$\mathrm{0SF} \longrightarrow \mathrm{2PM}, \qquad \mathrm{1SF} \longrightarrow \mathrm{4PM}, \qquad \mathrm{2SF} \longrightarrow \mathrm{6PM}, \ldots$$

  - Relies on mass-exchange symmetry and the particular polynomial dependence of the PM coefficients on the component masses.

Scatter angle correction [Barack & Long 2209.03740]

- 1SF correction expressed as integral of SF along background geodesic

$$\chi^{1\mathrm{SF}} = \int_{-\infty}^{+\infty} A_\alpha(\tau; b, v) F^\alpha(\tau) d\tau.$$

- We can split the self-force into *conservative* and *dissipative* pieces,

$$F^\alpha_{\mathrm{cons}}(\tau) = \frac{1}{2}\left[F^\alpha(\tau) + \epsilon^\alpha F^\alpha(-\tau)\right], \quad F^\alpha_{\mathrm{diss}}(\tau) = \frac{1}{2}\left[F^\alpha(\tau) - \epsilon^\alpha F^\alpha(-\tau)\right],$$

where $\epsilon^\alpha = (-1, 1, 1, -1)$ in Schwarzschild coords.

- Consider their effects separately,

$$\chi^{1\mathrm{SF}}_{\mathrm{cons}} = \int_0^{+\infty} A^{\mathrm{cons}}_\alpha(\tau; b, v) F^\alpha_{\mathrm{cons}}(\tau) d\tau$$

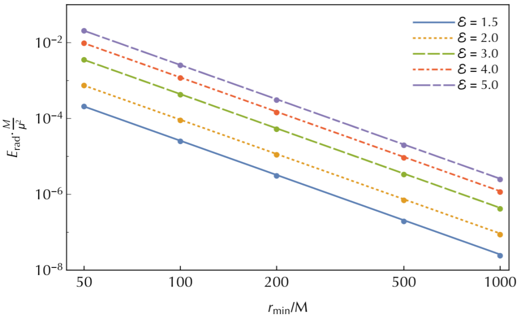
$$\chi^{1\mathrm{SF}}_{\mathrm{diss}} = \int_0^{+\infty} A^{\mathrm{diss}}_\alpha(\tau; b, v) F^\alpha_{\mathrm{diss}}(\tau) d\tau = \frac{1}{2}\left[\alpha_E E \cdot E_{\mathrm{rad}} + \alpha_L L \cdot L_{\mathrm{rad}}\right]$$

Gravitational scattering: dissipative sector

- Hopper & Cardoso studied the GW energy flux from a point mass moving along a geodesic scattering trajectory in the Schwarzschild spacetime. [Hopper & Cardoso 1706.02791]

- Did not compute the local self-force acting on the particle.

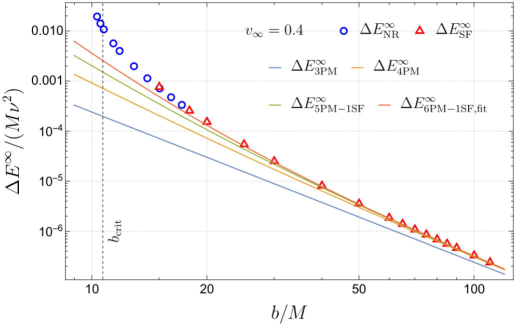
[Image credit: Hopper & Cardoso 1706.02791]



Gravitational scattering: dissipative sector

- Hopper & Cardoso's method has been recently replicated by Warburton. [Warburton 2512.02274]
- Performed additional comparisons between radiated energy estimates from SF, PM and numerical relativity.
- Similar calculation of the radiated angular momentum would allow a calculation of χ^{1SF}_{diss}.

[Image credit: Warburton 2512.02274]



- Dissipative calculations can use asymptotic quantities: conservative calculations need the local self-force.

Scalar-field toy model

- Toy model: scalar charge Q with mass m moving in a background Schwarzschild spacetime of mass M:

$$\nabla^\mu \nabla_\mu \Phi = -4\pi Q \int_{-\infty}^{+\infty} \frac{\delta^4(x - x_p(\tau))}{\sqrt{-g(x)}} d\tau. \tag{1}$$

- Scalar-field calculation captures the main challenges of gravitational self-force calculations, in a simpler overall framework.

- Parameter $q_s := Q^2/(mM) \ll 1$ takes the role of the mass ratio.

- Scalar self-interaction modifies trajectory according to:

$$m \frac{Du^\alpha}{d\tau} = Q \left(g^{\alpha\beta} + u^\alpha u^\beta \right) \nabla_\beta \Phi^R \tag{2}$$

Numerical self-force calculations

Decompose the field into spherical harmonics centred on the Schwarzschild black hole:

$$\Phi(t, r, \theta, \phi) = \frac{1}{r} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} \psi_{\ell m}(t, r) Y_{\ell m}(\theta, \phi)$$

Self-force can be computed using mode-sum regularisation [Barack and Ori gr-qc/9912010]:

$$F_\alpha(\tau) = \sum_{\ell=0}^{\infty} \left\{ \nabla_\alpha \left[\frac{1}{r} \sum_{m=-\ell}^{+\ell} \psi_{\ell m} Y_{\ell m} \right]_{x_p^\pm(\tau)} \mp \left(\ell + \frac{1}{2} \right) \underbrace{A_\alpha(\tau) - B_\alpha(\tau)}_{\text{regularisation parameters}} \right\}$$

Time-domain

- Solve $(1+1)d$ PDEs for $\psi_{\ell m}(t, r)$.
- Scattering: [Long & Barack 2209.03740]

Frequency-domain

- Construct $\psi_{\ell m}(t, r)$ from frequency-modes, obtained by solving ODEs.
- Scattering: [Whittall & Barack 2305.09724]

Direct reconstruction

- Frequency-modes $\psi_{\ell m\omega}$ obey ODE

$$\frac{d^2\psi_{\ell m\omega}}{dr_*^2} - \left[V_\ell(r) - \omega^2\right]\psi_{\ell m\omega} = S_{\ell m\omega}(r)$$

- Retarded solution expressed in terms of homogeneous solution basis $\psi^{\pm}_{\ell\omega}(r)$ using variation of parameters:

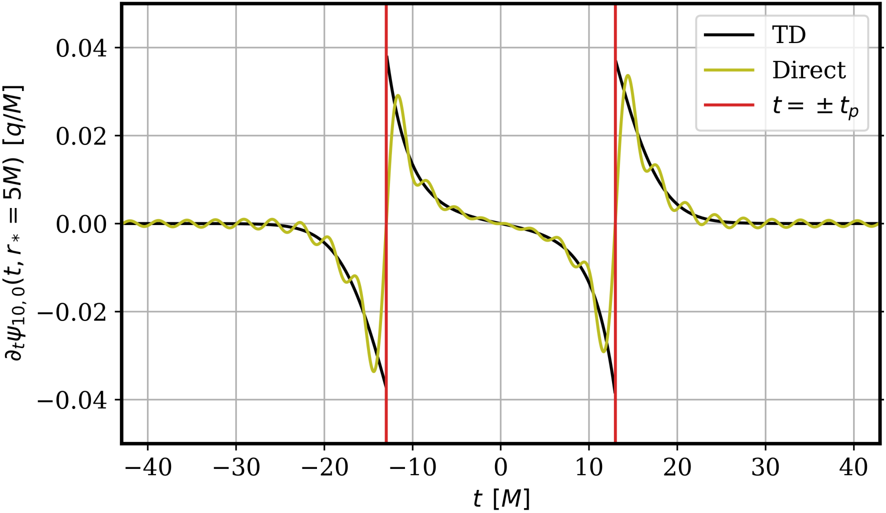
$$\psi_{\ell m\omega}(r) = \psi^{+}_{\ell\omega}(r)\int_{r_{\min}}^{r} \frac{\psi^{-}_{\ell\omega}(r')S_{\ell m\omega}(r')}{W_{\ell\omega}f(r')}dr' + \psi^{-}_{\ell\omega}(r)\int_{r}^{+\infty} \frac{\psi^{+}_{\ell\omega}(r')S_{\ell m\omega}(r')}{W_{\ell\omega}f(r')}dr'$$

- Direct reconstruction:

$$\psi_{\ell m}(t,r) \approx \Psi_{\ell m}(t,r;\omega_{\max}) := \int_{-\omega_{\max}}^{+\omega_{\max}} \psi_{\ell m\omega}(r)e^{-i\omega t}d\omega$$

is **not practical** due to the Gibbs phenomenon.

Direct reconstruction



EHS reconstruction [Barack, Ori & Sago 2008]

- EHS reconstruction: recover $\psi_{\ell m}(t,r)$ separately in $r \leq r_p(t)$ and $r \geq r_p(t)$ using homogeneous solutions.
- For example, field modes in the "internal" region $r \leq r_p(t)$ reconstructed from

$$\tilde{\psi}^{-}_{\ell m\omega}(r) := \psi^{-}_{\ell\omega}(r)\int_{r_{\min}}^{+\infty}\frac{\psi^{+}_{\ell\omega}(r')S_{\ell m\omega}(r')}{W_{\ell\omega}f(r')}dr'.$$

- Restores exponential, uniform convergence.
- External reconstruction: EHS cannot be applied in $r > r_p(t)$ for scatter orbits.
- Cancellation problem: significant loss of precision at high eccentricities [van de Meent 2016]. Limited to $r_p \sim r_{\min}$ for scatter orbits.

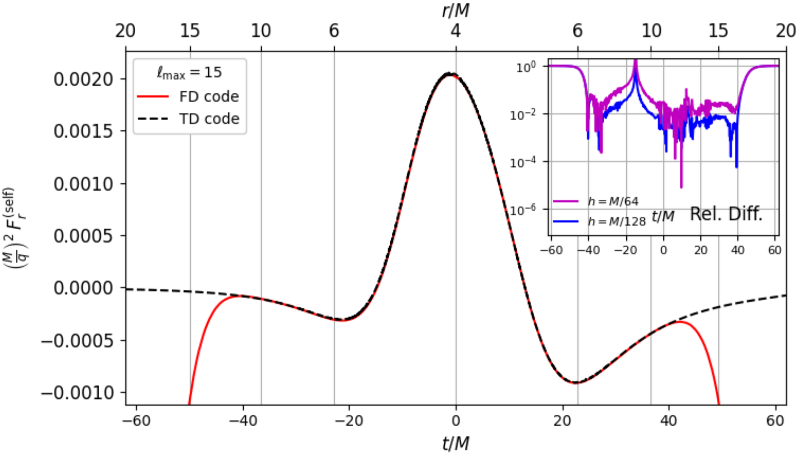
Self-force scattering in the frequency-domain

Scattering presents novel problems for FD self-force calculations

- Continuous frequency spectrum.
  - More frequency modes need to be computed.
  - Continuous spectra well approximated by discrete sampling.

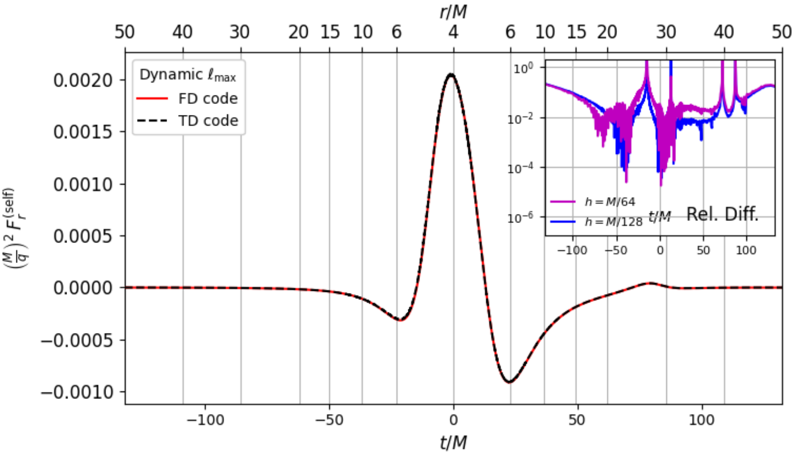
- Normalisation integrals are highly oscillatory and converge slowly.
  - Iterated source approach [Hopper & Cardoso 1706.02791]
    - $\rightarrow$ Currently only formulated for Schwarzschild.
  - Integration by parts [CW & Barack 2305.09724]

Self-force ($\ell_{\max} = 15$) [CW & Barack 2305.09724]



Good agreement with TD code near periapsis. Rapid deterioration in FD code as r increased.

Self-force (dynamic $\ell_{\max}$) [CW & Barack 2305.09724]



Cancellation forces us to truncate the ℓ-mode sum earlier and earlier as we move out along the orbit...

Beyond EHS...

- Cancellation between EHS frequency modes limits applicability of FD methods to points close to orbital turning points.
  - Highly-eccentric bound-orbit calcs make use of arbitrary-precision arithmetic to handle cancellation: impractical for scattering.

- Demonstrated accurate reconstruction of the scalar-field near the worldline from the inhomogeneous field in [CW, Barack, Long 2509.19439].

- Method relies on reprojection onto basis of Gegenbauer polynomials $\{C_k^\lambda(s)\}$ orthogonal on $[-1,1]$ wrt weighted inner product,

$$\int_{-1}^{+1}(1-s^2)^{\lambda-1/2}C_n^\lambda(s)C_m^\lambda(s)ds = h_n^\lambda\delta_{nm}.$$

  - Generalisation of Legendre ($\lambda = 1/2$) and Chebyshev ($\lambda = 0, 1$) polynomials.

Gegenbauer reconstruction [Gottlieb & Shu 1992 (et al), 1994, 1995, 1997]

Suppose $\psi_{\ell m}(t,r)$ is analytic (at fixed r) on interval $a \leq t \leq b$:

1. Compute the partial Fourier integrals using the inhomogeneous modes $\psi_{\ell m \omega}(r)$ for $t \in [a,b]$:

$$\Psi_{\ell m}(t,r;\omega_{\max}) := \int_{-\omega_{\max}}^{+\omega_{\max}} \psi_{\ell m \omega}(r) e^{-i\omega t} d\omega.$$

2. Project onto the Gegenbauer basis:

$$g_k^\lambda(r;\omega_{\max}) := \frac{1}{h_k^\lambda} \int_{-1}^{1} (1-s^2)^{\lambda - 1/2} \Psi_{\ell m}\left(t(s), r; \omega_{\max}\right) C_k^\lambda(s) ds.$$

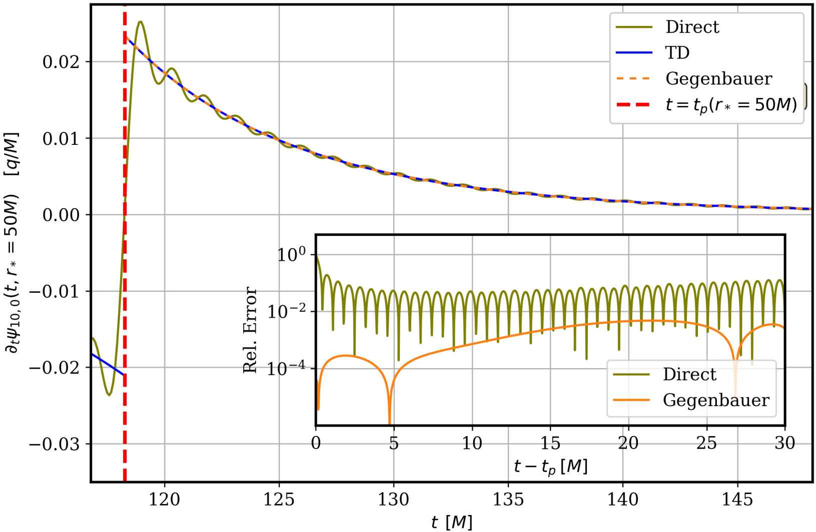
where $t(s) = \left[(b-a)s + (a+b)\right]/2$.

3. Approximate

$$\psi_{\ell m}(t,r) \approx \sum_{k=0}^{N} g_k^\lambda C_k^\lambda(s(t)).$$

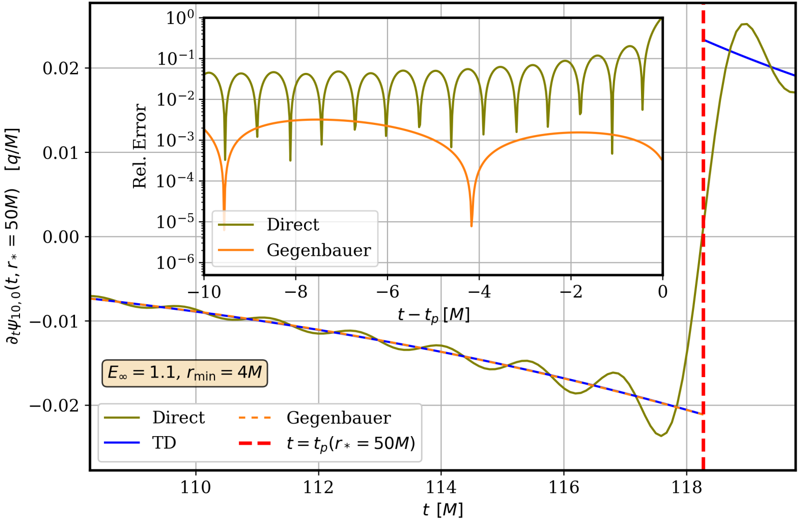
Gegenbauer approximant converges uniformly and exponentially on $a \leq t \leq b$, provided N, λ and $\omega_{\max} \to \infty$ in linear proportion.

Internal reconstruction: $r \leq r_p(t)$ [CW, Barack and Long 2509.19439]



Gegenbauer reconstruction effective and outperforms Direct reconstruction.

External reconstruction: $r \geq r_p(t)$ [CW, Barack and Long 2509.19439]



Gegenbauer reconstruction enables calculations in $r > r_p(t)$.

Gegenbauer reconstruction: discussion

- Circumvents Gibbs phenomenon and EHS challenges (cancellation, external reconstruction).
- Choice of N, λ impacts accuracy of approximant.
  - Robust parameter selection is key challenge for implementation.
- Computational cost: more integrals!
  - Gegenbauer reconstruction *will* be more expensive than EHS approach, but probably still feasible.
- Other approaches exist! Large body of under-exploited literature on the Gibbs phenomenon.

Exploitation

With the self-force scatter data, we can...

- Calibrate the conservative EOB Hamiltonian.
- Cross-validate with PM series [Barack et al 2304.09200] and numerical relativity etc.
- Extract unknown PM terms [Warburton 2512.02274].
- Resum PM results using SF information from the $b \to b_c(v)$ limit, extending the former's range of validity. [Long, CW and Barack 2406.08363, Barack et al 2602.10089]
  - Relativistic beaming enhances large-ℓ modes near to periapsis in the $v \to 1$, $b \to b_c(v)$ limit, where FD method excels.
- Deduce bound-orbit observables in the SF expansion using nascent self-force B2B mappings. [Gonzo and Shi 2304.06066, Gonzo, Lewis and Pound 2409.03437].

Conclusions and outlook

- Hyperbolic scattering is now a firmly-established area of self-force research.
- Frequency-domain methods adequate in the dissipative sector (at least for Schwarzschild.)
- Frequency-domain methods relying on EHS struggle in the conservative sector for highly eccentric orbits (especially scattering).
  - Reconstruction directly from inhomogeneous modes may be possible.
- Conservative gravitational SF calculations under development in both the time-domain [Vaswani, Long, Barack, ...] and frequency-domain [Warburton, Cunningham, Dolan, Seenivasan, Barack,...].

Bonus slide: Resummation

- Log divergence in $\chi^{0\mathrm{SF}}$:

$$\chi^{0SF} \sim A_0(v) \log\left(\frac{\delta b}{b_c(v)}\right) \text{ as } b \to b_c(v),$$

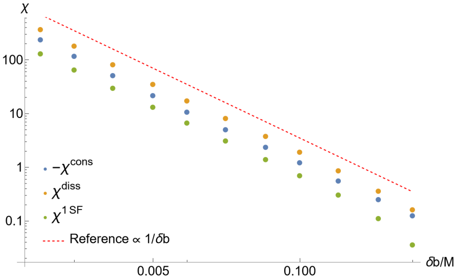
where, recall, $\delta b := b - b_c(v)$, and

$$A_0(v) = -\left(1 - \frac{12M^2(1-v^2)}{v^2 b_c(v)^2}\right)^{1/2}.$$

- Faster divergence at 1SF,

$$\chi^{1SF} \sim A_1(v) \frac{b_c(v)}{\delta b},$$

as $b \to b_c(v)$.



Bonus slide: Resummation

- Introduce

$$\Delta\chi(v,b) := A_0\left[\log\left(1-\frac{b_c(v)(1-\epsilon A_1/A_0)}{b}\right)+\sum_{k=1}^{4}\frac{1}{k}\left(\frac{b_c(v)(1-\epsilon A_1/A_0)}{b}\right)^k\right].$$

  - $\Delta\chi = O(b^{-5})$ as $b\to\infty$
  - Matches the $b\to b_c(v)$ divergences of $\chi(v,b)$ at both 0SF and 1SF.

- Resummed scatter angle:

$$\tilde{\chi}(v,b) := \chi_{4\mathrm{PM}}(v,b)+\Delta\chi(v,b).$$

  - Matches $b\to\infty$ behaviour of χ through 4PM order.
  - Matches $b\to b_c(v)$ behaviour at 0SF and 1SF.

- Similar to geodesic order approach introduced in [Damour & Rettegno 2211.01399], but extended to 1SF.

Bonus slide: Resummation [Long, CW and Barack 2406.08363]

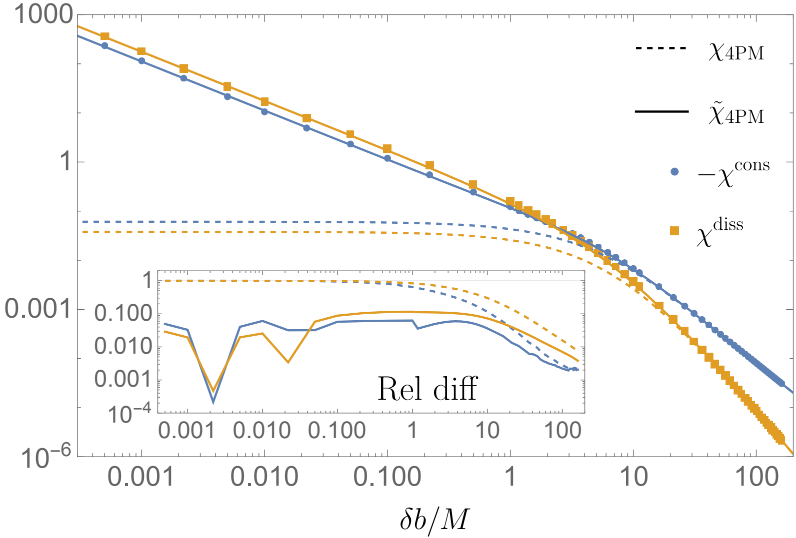


Figure: the resummation procedure significantly improves agreement with the numerical SF data, even in the weak-field. ($v = 0.5$)