Black hole scattering: the self-force approach

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STAG RESEARCH CENTRE

Talk outline

A) Introduction to self-force

B) Self-force in black hole scattering

C) Frequency-domain appproach

D) Ongoing work

PART A: Introduction to self-force

Reviews:

L. Barack & A. Pound, *Self-force and radiation reaction in general relativity*, 2019 Rep. Prog. Phys. **82** 016904 [arXiv:1805.10385]

E. Poisson, A. Pound & I. Vega, *The Motion of Point Particles in Curved Spacetime*, Living Rev. Relativ. **14**, 7 (2011) [arXiv:1102.0529]

The 2-body problem in GR: approaches

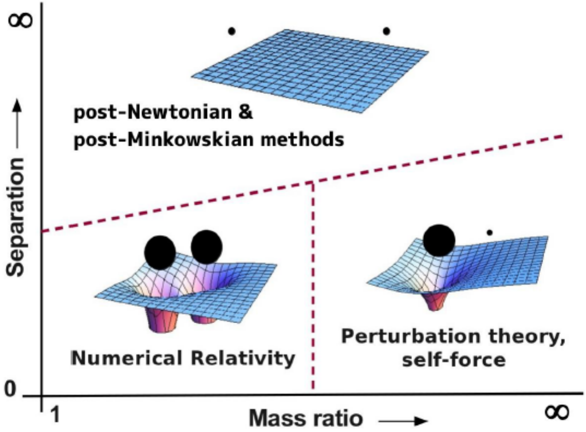
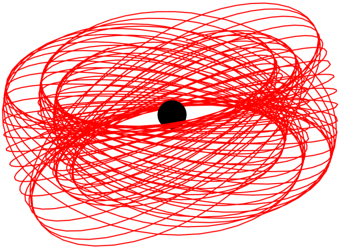


Image credit: L. Barack & A. Pound

Extreme mass ratio inspirals (EMRIs)

Created using KerrGeodesics package from BHP toolkit.



- Highly asymmetric compact binaries. Typical mass ratios

$$q \sim \frac{10 M_\odot}{10^6 M_\odot} = 10^{-5} \ll 1 \quad (1)$$

- Inspiral slow compared to orbital periods:

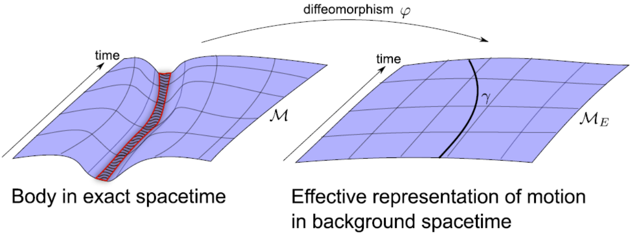
$$T_{\mathrm{RR}} \sim T_{\mathrm{orb}}/q \gg T_{\mathrm{orb}}. \quad (2)$$

- Large number of gravitational wavecycles in LISA band before merger:

$$N_{\mathrm{orb}} \sim 1/q \sim 10^5. \quad (3)$$

- Orbital dynamics complicated. Geodesics tri-periodic and generically ergodic.
- EMRIs offer a precision probe of strong-field geometry around black-holes.

Self-force principles



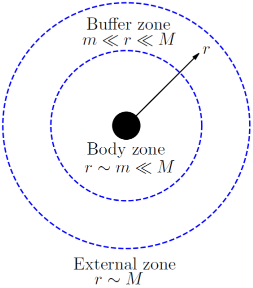


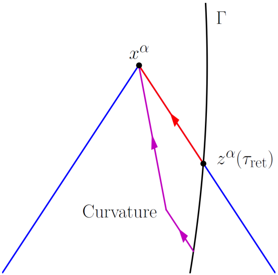
Image credit: A. Pound

- Motion of small body given effective representation in background spacetime of the larger object.
- No need for ad hoc regularisation procedures. EOM derived using *matched asymptotic expansions* [Mino, Sasaki & Tanaka 1997].
- No need to *assume* a point-particle description; effective point-particle description is *derived*.

1SF equation of motion

- Metric perturbation split into "direct" and "tail" contributions:

$$g_{\alpha\beta}^{\text{phys}} = g_{\alpha\beta} + h_{\alpha\beta}^{\text{direct}} + h_{\alpha\beta}^{\text{tail}}. \qquad (4)$$



- Only $h_{\alpha\beta}^{\text{tail}}$ contributes to the self-force:

$$m\frac{D^2 z^\alpha}{d\tau^2} = m\nabla^{\alpha\beta\gamma} h_{\beta\gamma}^{\text{tail}}\Big|_{z(\tau)} =: F_{\text{self}}^\alpha, \qquad (5)$$

where

$$\nabla^{\alpha\beta\gamma} h_{\gamma\beta} := -\frac{1}{2}\left(g^{\alpha\beta} + u^\alpha u^\beta\right) u^\gamma u^\delta \left(2\nabla_\delta h_{\beta\gamma} - \nabla_\beta h_{\gamma\delta}\right). \qquad (6)$$

Computational approach: mode-sum regularisation

- Singular field subtracted mode-by-mode in a spherical harmonic expansion around the large BH:

$$F_{\rm self}(\tau) = m\sum_{\ell=0}^{\infty}\left[\left(\nabla h^{\rm ret}\right)^{\ell} - \left(\nabla h^{\rm direct}\right)^{\ell}\right]_{z(\tau)} \tag{7}$$
$$= \sum_{\ell=0}^{\infty}\left[m\left(\nabla h^{\rm ret}\right)^{\ell}\Big|_{z(\tau)} - A(z)\ell - B(z) - C(z)/\ell\right] - D(z).$$

- Regularization parameters: derived analytically for generic Kerr orbits. [Barack & Ori 2000-03]

- Numerical input: modes of $h^{\rm ret}_{\alpha\beta}$ calculated numerically by solving perturbation equations with point-particle source and retarded BCs.

- Can accelerate convergence by subtracting additional parameters.

PART B: Self-force in black hole scattering

L. Barack & O. Long, *Self-force correction to the deflection angle in black-hole scattering: A scalar charge toy model*, Phys. Rev. D **106** 104031 (2022) [arXiv:2209.03740].

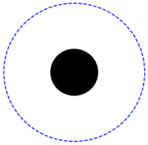
L. Barack, Z. Bern, E. Herrmann, O. Long, J. Parra-Martinez & R. Roiban, M. Ruf, C. Shen, M. Solon, F. Teng & M. Zeng, *Comparison of post-Minkowskian and self-force expansions: Scattering in a scalar charge toy model*, Phys. Rev. D **108** 024025 (2023) [arXiv:2304.09200].

Scatter orbits

Particle starts at radial infinity at early times with velocity v and *impact parameter* b:

$$b = \lim_{\tau \to -\infty} r_p(\tau) \sin \left| \varphi_p(\tau) - \varphi_p(-\infty) \right| . \tag{8}$$

Provided $b > b_{\mathrm{c}}(v)$, particle scatters off central black hole, approaching to within periapsis distance r_{min}.



Why study scattering?

- Theoretical grounds:
  1. Can probe sub-ISCO region even at low velocities; down to light ring $r = 3M$ with large v.
  2. Scattering angle $\chi(b, v)$ defined unambiguously, even with radiation.

- Boundary-to-bound relations between scatter and bound orbit observables, derived using effective-field-theory. [Kalin & Porto 2020]

- $\chi_{\rm 1SF}$ determines **full** conservative dynamics to 4PM, valid at *any* mass ratio. Extend to 6PM with $\chi_{\rm 2SF}$ [Damour 2020]. PM expansion of χ can be used to calibrate effective-one-body models [Damour 2016].

- Can compare SF results with analytical PM for mutual validation; benchmark/calibrate PM in strong-field.

Self-force corrections to the scatter angle

- The self-force correction is defined by

$$\delta\chi := \chi(b, v) - \chi_0(b, v) = O(q), \tag{9}$$

where $\chi_0 := \lim_{q \to 0} \chi$ is the scatter angle of the geodesic with the same (b, v).

- Correction expressed as integral over the worldline, [Barack & Long 2022]

$$\delta\chi = \int_{-\infty}^{+\infty} A_\alpha(\tau; b, v) F^\alpha_{\mathrm{self}}(\tau) d\tau. \tag{10}$$

At $O(q)$, integral may be evaluated along limiting geodesic.

Conservative and dissipative effects

We can split the self-force into *conservative* and *dissipative* pieces,

$$F^{\alpha}_{\rm cons} = \frac{1}{2}\left[F^{\alpha}_{\rm self}(h^{\rm ret}) + F^{\alpha}_{\rm self}(h^{\rm adv})\right], \tag{11}$$

$$F^{\alpha}_{\rm diss} = \frac{1}{2}\left[F^{\alpha}_{\rm self}(h^{\rm ret}) - F^{\alpha}_{\rm self}(h^{\rm adv})\right], \tag{12}$$

and consider their effects separately, [Barack & Long 2022]

$$\delta\chi_{\rm cons} = \int_0^{+\infty} A^{\rm cons}_{\alpha}(\tau; b, v) F^{\alpha}_{\rm cons}(\tau) d\tau, \tag{13}$$

$$\delta\chi_{\rm diss} = \int_0^{+\infty} A^{\rm diss}_{\alpha}(\tau; b, v) F^{\alpha}_{\rm diss}(\tau) d\tau. \tag{14}$$

Scalar-field toy model in Schwarzschild

- Toy model: scalar charge Q with mass m moving in a background Schwarzschild spacetime of mass M:

$$\nabla^\mu \nabla_\mu \Phi = -4\pi Q \int_{-\infty}^{+\infty} \frac{\delta^4(x - z(\tau))}{\sqrt{-g(x)}} d\tau, \tag{15}$$

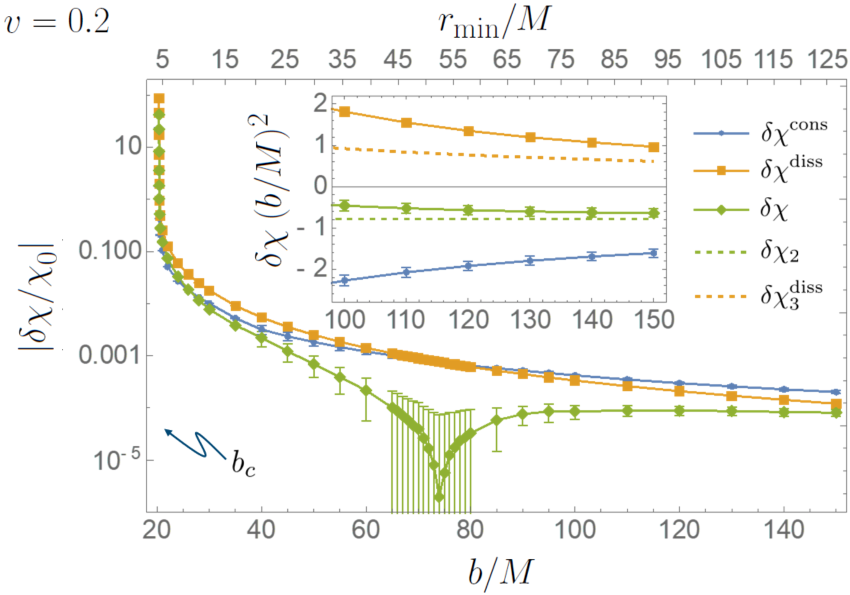
$$\frac{Dp^\alpha}{d\tau} = Q \nabla^\alpha \Phi^{\text{tail}} =: F^\alpha_{\text{self}}. \tag{16}$$

- Scalar-field calculation captures the main challenges of gravitational self-force calculations, in a simpler overall framework.

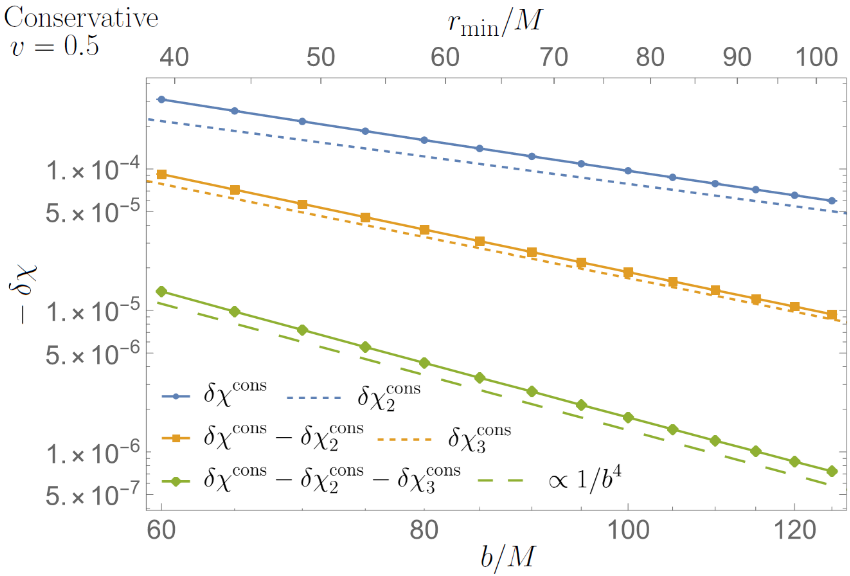
- Parameter $q_s := Q^2/(mM) \ll 1$ takes the role of the mass ratio. Integral formulae for $\delta\chi$ essentially unchanged.

- First numerical calculations by Long & Barack using their (1+1)D time-domain code for the self-force.

Early scatter angle results [Barack & Long 2022]



PM comparisons [Barack et al 2023]



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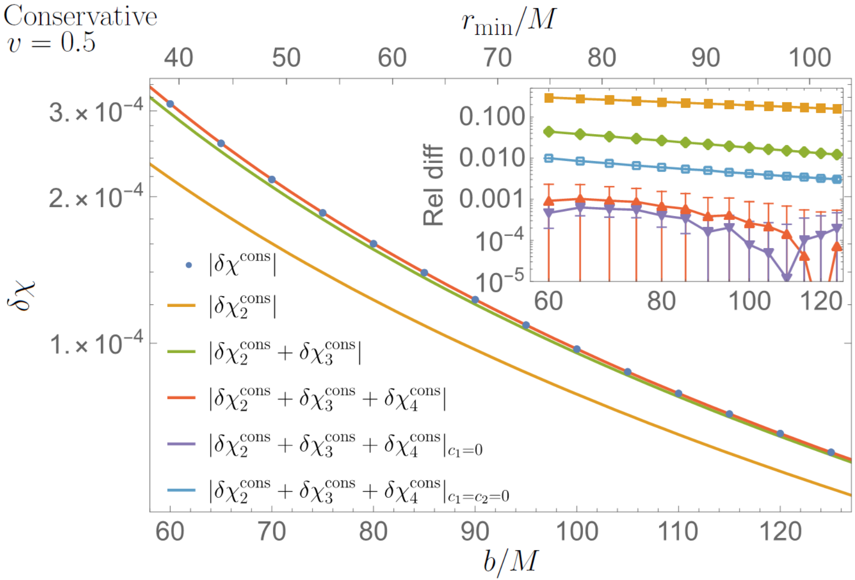


PART C: Frequency-domain approach

C. Whittall & L. Barack, *Frequency-domain approach to self-force in hyperbolic scattering*, Phys. Rev. D **108** 064017 (2023) [arXiv:2305.09724].

Frequency-domain methods

- Fields are additionally decomposed into Fourier harmonics, e.g.

$$\psi_{\ell m}(t, r) = \int_{-\infty}^{+\infty} \psi_{\ell m\omega}(r) e^{-i\omega t}. \tag{17}$$

- Many frequency-domain (FD) self-force codes in existence for bound orbits. Valued for their accuracy and efficiency.

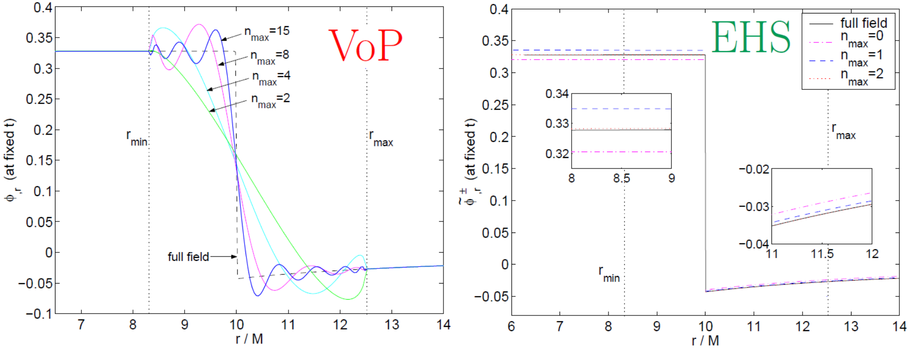
- FD methods expected to retain these advantages when moving to unbound orbits, but challenges must be overcome:
  - Continuous spectrum.
  - Source with non-compact radial support.
  - Cancellation during TD reconstruction.

We use a scalar-field toy model in Schwarzschild to investigate and manage these problems.

Extended homogeneous solutions [Barack, Ori & Sago 2008]

- Gibbs phenomenon: impractical to reconstruct SF modes from physical inhomogeneous solution $\psi_{\ell m\omega}(r)$.

- Method of Extended Homogeneous Solutions restores exponential, uniform convergence.



Extended homogeneous solutions: unbound orbits

- Physical time-domain field is reconstructed piecewise from **homogeneous** solutions.

- For example, SF modes in the “internal” region $r \leq r_p(t)$ reconstructed from

$$\tilde{\psi}^{-}_{\ell m\omega}(r) := C^{-}_{\ell m\omega}\psi^{-}_{\ell\omega}(r), \tag{18}$$

  where normalisation the factor $C^{-}_{\ell m\omega}$ is such that EHS and physical field coincide in $r \leq r_{\min}$.

- For unbound orbits, EHS cannot be used to reconstruct field in the “external” region $r > r_p(t)$.

We use EHS and one-sided mode-sum regularisation

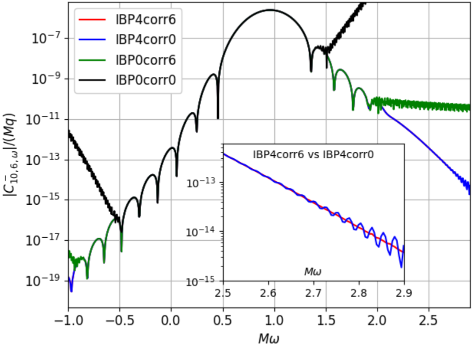
Truncation problem

- Normalisation factor $C^-_{\ell m\omega}$ can be expressed as an integral over the (unbounded) radial extent of the orbit:

$$C^-_{\ell m\omega} = \int_{r_{\min}}^{+\infty} \frac{\psi^+_{\ell\omega}(r')S_{\ell m\omega}(r')}{W_{\ell\omega} f(r')} dr'. \qquad (19)$$

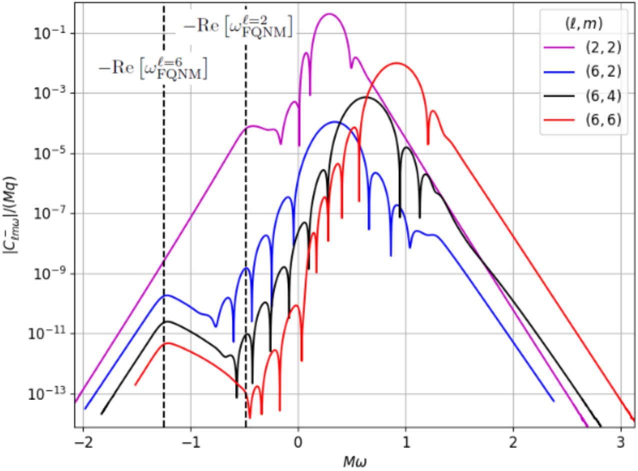
- Slow, oscillatory convergence: problems truncating at finite $r_{\max}$.

- Developed solutions:
  1. Tail corrections: use large-r approximation to integrand to derive analytical estimates to the neglected tail.
  2. Integration by parts (IBP): use IBP to increase decay rate of integrand.

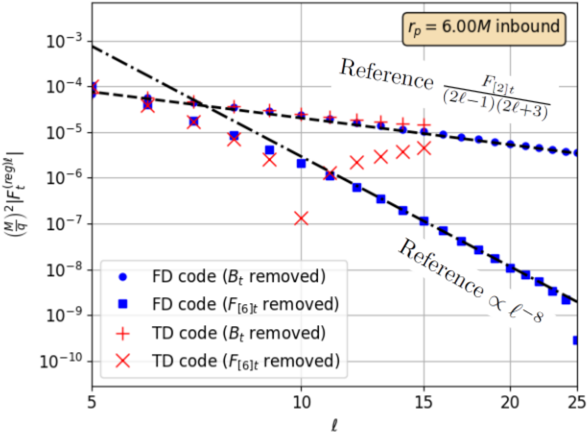


$C^-_{\ell m\omega}$ spectra

Example $C^-_{\ell m\omega}$ spectra for orbit $E = 1.1$, $r_{\min} = 4M$. Note QNM features.



Self-force: regularisation tests

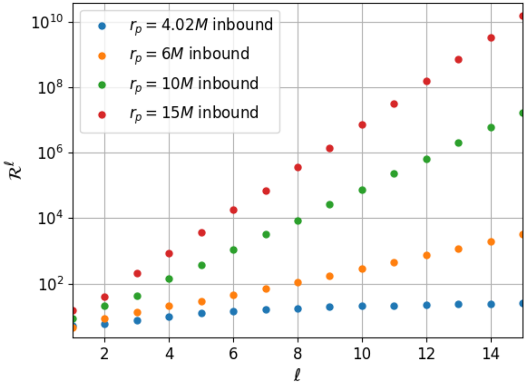


FD code agrees better with regularisation parameters at this radius

$$F_{\text{self}}(\tau) = \sum_{\ell=0}^{\infty} \left[q \left(\nabla \Phi^{\text{ret}} \right)^{\ell} \Big|_{z(\tau)} - A(z)\ell - B(z) - C(z)/\ell - H.O.P \right] - D(z)$$

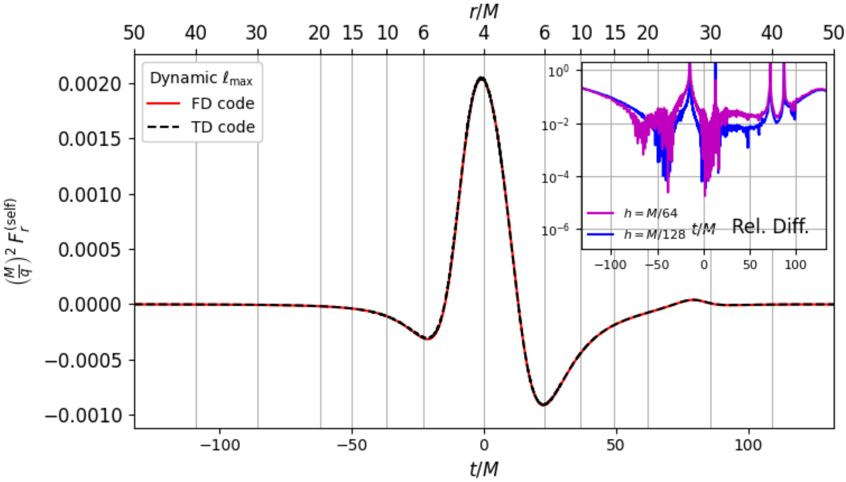
Cancellation problem

- Significant cancellation between low-frequency modes at large ℓ and r.
- Caused by unphysical growth of the EHS field.
- Problem intrinsic to EHS approach. Afflicts scatter calculations more severely than bound orbit case.



Partially mitigate using dynamic ℓ-truncation in the mode-sum.

Self-force: along orbit



Gradual loss of accuracy along orbit due to progressive loss of ℓ-modes.

PART D: Ongoing work

Analytical calculation at large r (preliminary)

- Supplement FD code with analytic expansion of the SF in $1/r$.
- Makes use of a hierarchical expansion, [Barack 1998]

$$\psi_{\ell m}(u,v) = \sum_{N=0}^{\infty} \psi_N(u,v), \tag{20}$$

$$\psi_{0,uv} + V_0(r)\psi_0 = S(u,v), \tag{21}$$

$$\psi_{N,uv} + V_0(r)\psi_N = -\delta V(r)\psi_{N-1} \quad (N>0), \tag{22}$$

where $V_0(r)$ approximates asymptotic behaviour of exact potential $V(r)$, and $\delta V(r) := V(r) - V_0(r)$.

- ψ_0 (complete) does not contribute to SF; ψ_1 (underway) gives leading large-r behaviour.

PM resummation (preliminary)

- As $b \to b_c(v)$,

$$\chi_0 \sim A(v) \log\left(1 - \frac{b_c(v)}{b}\right) + \text{const}(v), \quad \delta\chi_{1\text{SF}} \sim q_s B(v) \frac{b_c(v)}{b - b_c(v)}.$$

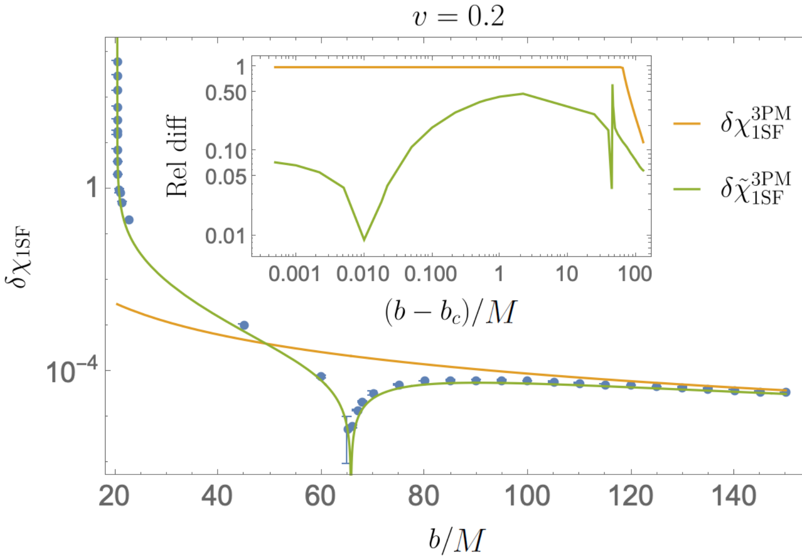
$A(v)$ known analytically; $B(v)$ inferred from SF calculations.

- Consider the function

$$\Psi^{n\text{PM}} = A\left[\log\left(1 - \frac{b_c(1 - q_s B/A)}{b}\right) + \sum_{k=1}^{n} \frac{1}{k}\left(\frac{b_c(1 - q_s B/A)}{b}\right)^k\right].$$

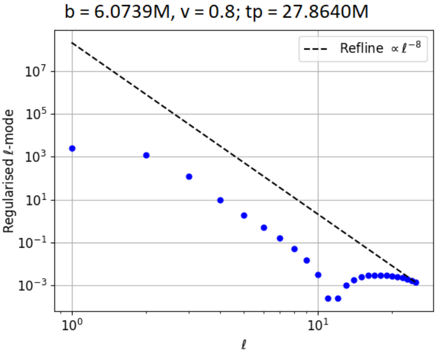
- We define the **resummed scatter angle** $\tilde{\chi}^{n\text{PM}} := \chi^{n\text{PM}} + \Psi^{n\text{PM}}$.
  - Agrees with $\chi^{n\text{PM}}$ through $n\text{PM}$ order.
  - Matches the 0SF and 1SF divergences near separatrix.

PM resummation (preliminary)



PM resummation: additional developments (preliminary)

- **High velocities:** large-ℓ modes become more important at higher velocities.
  - Possibly related to relativistic beaming of radiation.
  - Effect strongest near periapsis.
  - FD code can get $\ell \geq 15$ modes near periapsis.
  - Developing FD/TD hybrid method.



- **Direct approach**: express $B(v)$ as integral over critical orbit, $b = b_c(v)$.
  - Only need to calculate SF along critical orbit. More accurate and efficient than fitting.
  - FD approach needs to handle a discrete, distributional piece of the spectrum arising from asymptotic circular orbit.

Prospects

- Analytical results for SF at large r: useful for both TD and FD approaches.
- Improved TD methods also under development, including spectral methods with hyperboloidal slicing and compactification.
- Routes to gravity?
  - Direct Lorenz-gauge calculation [Ackay, Warburton, Barack 2013]
  - Radiation-gauge reconstruction [Pound, Merlin, Barack 2013]
  - Lorenz-gauge reconstruction [Dolan, Durkan, Kavanagh, Wardell 2023]
- Second order?
  - Easier than bound? No disparate timescales.
  - Would give conservative dynamics to 6PM.
  - Some way off.