Self-force in Hyperbolic Black Hole Scattering

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Talk outline

A) Introduction to self-force

B) Self-force in black hole scattering

C) Frequency-domain appproach

D) Future work

PART A: Introduction to self-force

The 2-body problem in GR: approaches

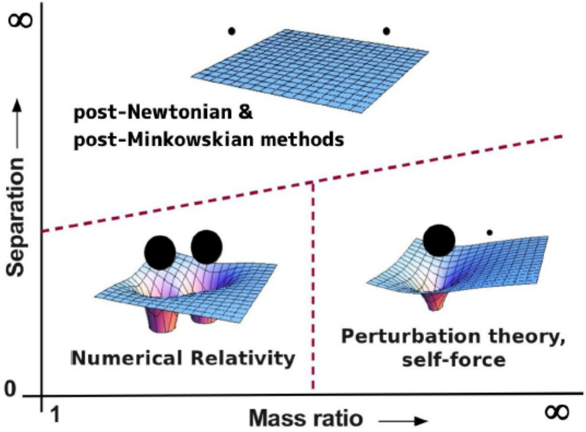
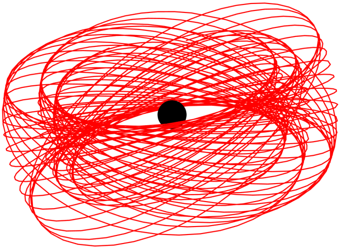


Image credit: L. Barack & A. Pound

Extreme mass ratio inspirals (EMRIs)

Created using KerrGeodesics package from BHP toolkit.



- Highly asymmetric compact binaries. Typical mass ratios

$$q \sim \frac{10 M_\odot}{10^6 M_\odot} = 10^{-5} \ll 1 \quad (1)$$

- Inspiral slow compared to orbital periods:

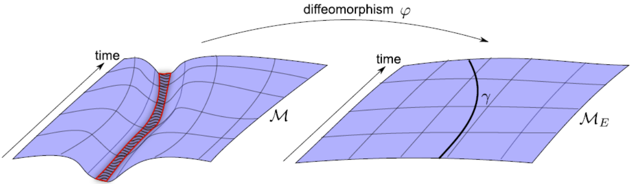
$$T_{\mathrm{RR}} \sim T_{\mathrm{orb}}/q \gg T_{\mathrm{orb}}. \quad (2)$$

- Large number of gravitational wavecycles in LISA band before merger:

$$N_{\mathrm{orb}} \sim 1/q \sim 10^5. \quad (3)$$

- Orbital dynamics complicated. Geodesics tri-periodic and generically ergodic.
- EMRIs offer a precision probe of strong-field geometry around black-holes.

Self-force principles



Body in exact spacetime

Effective representation of motion in background spacetime

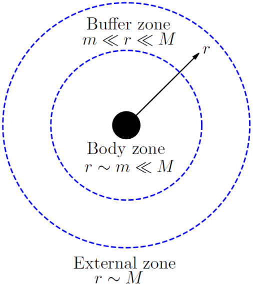
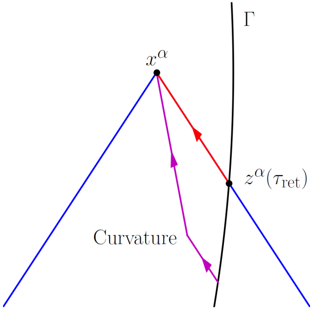


Image credit: A. Pound

- Motion of small body given effective representation in background spacetime of the larger object.
- No need for ad hoc regularisation procedures. EOM derived using *matched asymptotic expansions* [Mino, Sasaki & Tanaka 1997].
- No need to *assume* a point-particle description; effective point-particle description is *derived*.

1SF equation of motion



- Metric perturbation split into "direct" and "tail" contributions:

$$g^{\rm phys}_{\alpha\beta} = g_{\alpha\beta} + h^{\rm direct}_{\alpha\beta} + h^{\rm tail}_{\alpha\beta}. \tag{4}$$

- Only $h^{\rm tail}_{\alpha\beta}$ contributes to the self-force:

$$m\frac{D^2 z^\alpha}{d\tau^2} = m\nabla^{\alpha\beta\gamma} h^{\rm tail}_{\beta\gamma}\Big|_{z(\tau)} =: F^\alpha_{\rm self}, \tag{5}$$

where

$$\nabla^{\alpha\beta\gamma} h_{\gamma\beta} := -\frac{1}{2}\left(g^{\alpha\beta} + u^\alpha u^\beta\right) u^\gamma u^\delta \left(2\nabla_\delta h_{\beta\gamma} - \nabla_\beta h_{\gamma\delta}\right). \tag{6}$$

R-S decomposition [Detweiler & Whiting 2003]

- May also write

$$F^{\alpha}_{\rm self}(\tau) = m\nabla^{\alpha\beta\gamma} h^{\rm R}_{\beta\gamma}\Big|_{z(\tau)} = \lim_{x\to z(\tau)} m\nabla^{\alpha\beta\gamma}\left(h^{\rm ret}_{\beta\gamma}(x) - h^{\rm S}_{\beta\gamma}(x)\right) \quad (7)$$

where the regular and singular fields are defined in terms of acausal Green's functions.

- Unlike $h^{\rm tail}_{\alpha\beta}$, the regular field is a **vacuum solution** of the linearised Einstein equations. The singular field represents the small object's self-field and does not contribute to the SF.

- Motion with 1st order self-force is equivalent to geodesic motion in the effective metric

$$g^{\rm eff}_{\alpha\beta} = g_{\alpha\beta} + h^{\rm R}_{\alpha\beta}. \quad (8)$$

Computational approaches I: mode-sum regularisation

- Singular field subtracted mode-by-mode in a spherical harmonic expansion around the large BH:

$$F_{\mathrm{self}}(\tau) = m \sum_{\ell=0}^{\infty} \left[\left(\nabla h^{\mathrm{ret}}\right)^{\ell} - \left(\nabla h^{\mathrm{S}}\right)^{\ell} \right]_{z(\tau)} \tag{9}$$
$$= \sum_{\ell=0}^{\infty} \left[m \left(\nabla h^{\mathrm{ret}}\right)^{\ell} \Big|_{z(\tau)} - A(z)\ell - B(z) - C(z)/\ell \right] - D(z).$$

- Regularization parameters: derived analytically for generic Kerr orbits. [Barack & Ori 2000-03]

- Numerical input: modes of $h^{\mathrm{ret}}_{\alpha\beta}$ calculated numerically by solving perturbation equations with point-particle source and retarded BCs.

Computational approaches II: effective sources [Barack & Golbourn 2007, Vega & Detweiler 2008,...]

- **Puncture field** $h^{\mathcal{P}}_{\alpha\beta} \approx h^{S}_{\alpha\beta}$ constructed analytically, such that

$$F_{\mathrm{self}}(\tau) = m\nabla h^{\mathcal{R}}\Big|_{z(\tau)}, \tag{10}$$

where $h^{\mathcal{R}} := h^{\mathrm{ret}} - h^{\mathcal{P}}$ is the **residual field**.

- Linearised field equation $\delta G_{\mu\nu}[h] = T_{\mu\nu}$ is rewritten:

$$\delta G_{\mu\nu}\left[h^{\mathcal{R}}\right] = T_{\mu\nu} - \delta G_{\mu\nu}\left[h^{\mathcal{P}}\right] =: S^{\mathrm{eff}}_{\mu\nu} \tag{11}$$

and solved numerically.

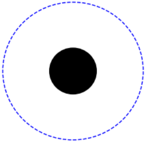
PART B: Self-force in black hole scattering

Scatter orbits

Particle starts at radial infinity at early times with velocity v and *impact parameter* b:

$$b = \lim_{\tau \to -\infty} r_p(\tau) \sin |\varphi_p(\tau) - \varphi_p(-\infty)| \,. \tag{12}$$

Provided $b > b_{\text{crit}}(v)$, particle scatters off central black hole, approaching to within periapsis distance $r_{\min}$.



Why study scattering?

- Theoretical grounds:
  1. Can probe sub-ISCO region even at low velocities; down to light ring $r = 3M$ with large v.
  2. Scattering angle $\chi(b, v)$ defined unambiguously, even with radiation.

- Boundary-to-bound relations between scatter and bound orbit observables, derived using effective-field-theory. [Kalin & Porto 2020]

- PM expansion of χ can be used to calibrate effective-one-body models [Damour 2016].

- $\chi_{1\mathrm{SF}}$ determines **full** conservative dynamics to 4PM, valid at *any* mass ratio. Extend to 6PM with $\chi_{2\mathrm{SF}}$ [Damour 2020].

Comparisons with PM

- Significant recent progress in PM theory, driven by techniques from outside the usual community (EFT, amplitudes).
- State of the art results at 4PM. [Bern et al 2021 and Dlapa et al 2022]
- Comparisons between PM and SF approaches allow mutual validation. [Barack et al 2023]
- SF results are "exact" i.e. contain PM terms of all orders at given order in q. Benchmark PM results in strong-field regime.

Self-force corrections to the scatter angle

- The self-force correction is defined by

$$\delta\chi := \chi(b, v) - \chi_0(b, v) = O(q), \tag{13}$$

where $\chi_0 := \lim_{q\to 0} \chi$ is the scatter angle of the geodesic with the same (b, v).

- Correction expressed as integral over the worldline, [Barack & Long 2022]

$$\delta\chi = \int_{-\infty}^{+\infty} A_\alpha(\tau; b, v) F^\alpha_{\mathrm{self}}(\tau) d\tau. \tag{14}$$

At $O(q)$, integral may be evaluated along limiting geodesic.

Conservative and dissipative effects

We can split the self-force into *conservative* and *dissipative* pieces,

$$F^{\alpha}_{\mathrm{cons}} = \frac{1}{2}\left[F^{\alpha}_{\mathrm{self}}(h^{\mathrm{ret}}) + F^{\alpha}_{\mathrm{self}}(h^{\mathrm{adv}})\right], \tag{15}$$

$$F^{\alpha}_{\mathrm{diss}} = \frac{1}{2}\left[F^{\alpha}_{\mathrm{self}}(h^{\mathrm{ret}}) - F^{\alpha}_{\mathrm{self}}(h^{\mathrm{adv}})\right], \tag{16}$$

and consider their effects separately, [Barack & Long 2022]

$$\delta\chi_{\mathrm{cons}} = \int_0^{+\infty} A^{\mathrm{cons}}_{\alpha}(\tau; b, v) F^{\alpha}_{\mathrm{cons}}(\tau) d\tau, \tag{17}$$

$$\delta\chi_{\mathrm{diss}} = \int_0^{+\infty} A^{\mathrm{diss}}_{\alpha}(\tau; b, v) F^{\alpha}_{\mathrm{diss}}(\tau) d\tau. \tag{18}$$

Scalar-field toy model in Schwarzschild

- Toy model: scalar charge Q with mass m moving in a background Schwarzschild spacetime of mass M:

$$\nabla^\mu \nabla_\mu \Phi = -4\pi Q \int_{-\infty}^{+\infty} \frac{\delta^4(x - z(\tau))}{\sqrt{-g(x)}} d\tau, \tag{19}$$

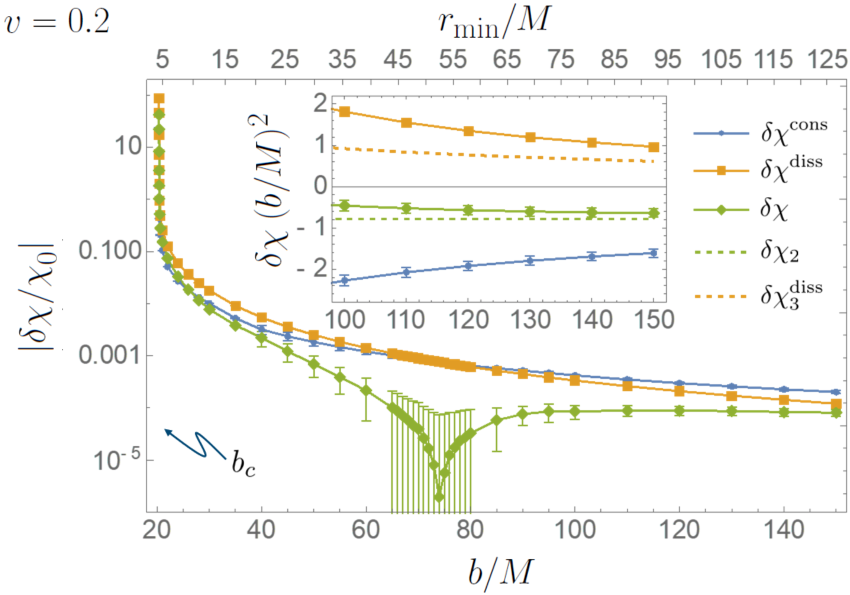
$$\frac{Dp^\alpha}{d\tau} = Q \nabla^\alpha \Phi =: F^\alpha_{\text{self}}. \tag{20}$$

- Scalar-field calculation captures the main challenges of gravitational self-force calculations, in a simpler overall framework.

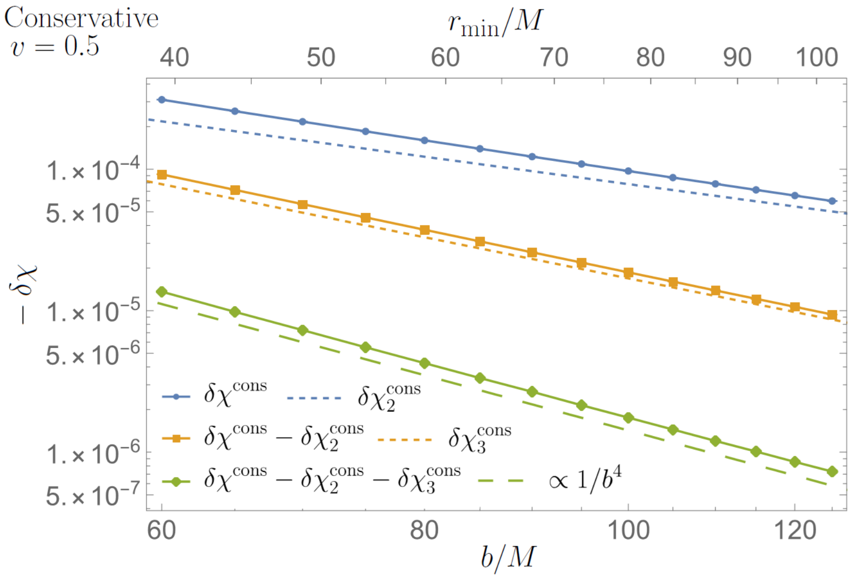
- Parameter $q_s := Q^2/(mM) \ll 1$ takes the role of the mass ratio. Integral formulae for $\delta\chi$ essentially unchanged.

- First numerical calculations by Long & Barack using their (1+1)D time-domain code for the self-force.

Early scatter angle results [Barack & Long 2022]



PM comparisons [Barack et al 2023]



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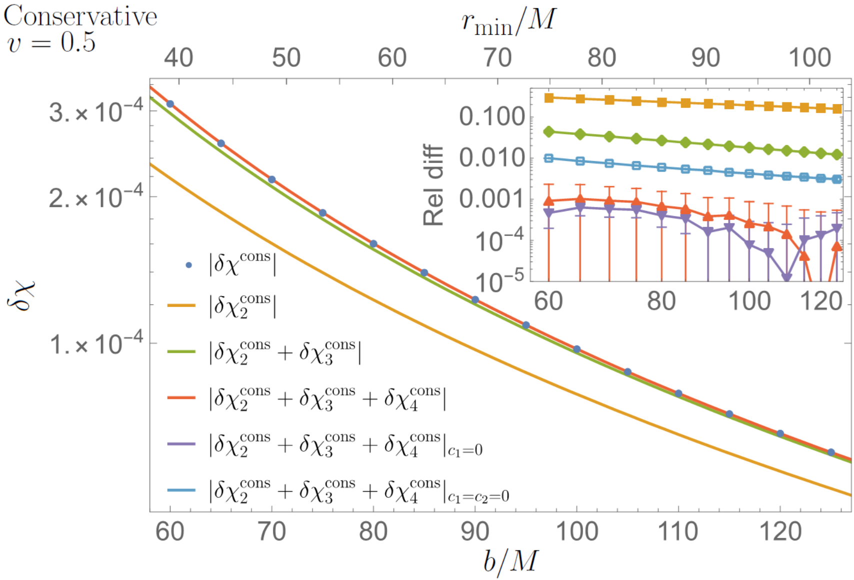


PART C: Frequency-domain approach

C. Whittall & L. Barack, *Frequency-domain approach to self-force in hyperbolic scattering*, Phys. Rev. D **108** 064017 (2023) [arXiv:2305.09724].

Frequency-domain methods

- Fields are additionally decomposed into Fourier harmonics, e.g.

$$\psi_{\ell m}(t,r) = \int_{-\infty}^{+\infty} \psi_{\ell m\omega}(r) e^{-i\omega t}. \tag{21}$$

- Many frequency-domain (FD) self-force codes in existence for bound orbits. Valued for their accuracy and efficiency.

- FD methods expected to retain these advantages when moving to unbound orbits, but challenges must be overcome:
  - Continuous spectrum.
  - Failure of EHS method.
  - Slowly convergent radial integrals.
  - Cancellation during TD reconstruction.

We use a scalar-field toy model in Schwarzschild to investigate and manage these problems.

Scalar-field toy model

- Field equation becomes

$$\frac{d^2\psi_{\ell m\omega}}{dr_*^2} - \left[V_\ell(r) - \omega^2\right]\psi_{\ell m\omega} = S_{\ell m\omega}(r). \tag{22}$$

- Admits homogeneous solutions $\psi^{\pm}_{\ell\omega}(r)$ obeying retarded BCs at either horizon or infinity. Retarded inhomogeneous solution constructed using variation of parameters:

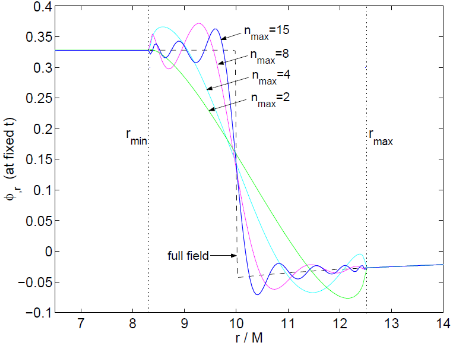
$$\psi_{\ell m\omega}(r) = \psi^{+}_{\ell\omega}(r)\int_{r_{\rm min}}^{r} \frac{\psi^{-}_{\ell\omega}(r')S_{\ell m\omega}(r')}{W_{\ell\omega}f(r')}dr' + \psi^{-}_{\ell\omega}(r)\int_{r}^{+\infty} \frac{\psi^{+}_{\ell\omega}(r')S_{\ell m\omega}(r')}{W_{\ell\omega}f(r')}dr' \tag{23}$$

- Gibbs phenomenon: impractical to reconstruct SF modes from physical solution $\psi_{\ell m\omega}(r)$.

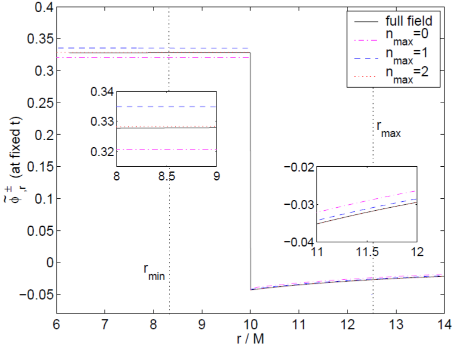
Extended homogeneous solutions [Barack, Ori & Sago 2008]

- Method of Extended Homogeneous Solutions restores exponential, uniform convergence.

VoP



EHS



Extended homogeneous solutions

- Physical time-domain field is reconstructed piecewise from **homogeneous** solutions. For example, SF modes in the "internal" region $r \leq r_p(t)$ reconstructed from

$$\tilde{\psi}^{-}_{\ell m\omega}(r) := \psi^{-}_{\ell\omega}(r) \int_{r_{\min}}^{+\infty} \frac{\psi^{+}_{\ell\omega}(r') S_{\ell m\omega}(r')}{W_{\ell\omega} f(r')} dr'. \tag{24}$$

- In vacuum region $r \leq r_{\min}$, this EHS field coincides with the physical, inhomogeneous field.

- For unbound orbits, EHS cannot be used to reconstruct field in the "external" region $r > r_p(t)$.

We use EHS and one-sided mode-sum regularisation

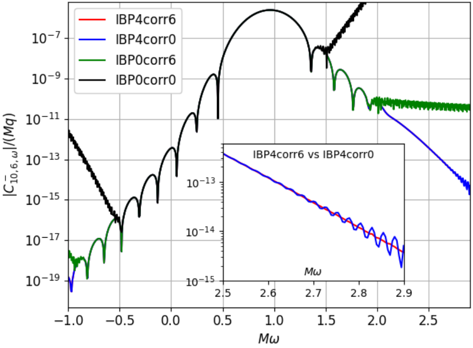
Truncation problem

- Need to evaluate the **normalisation integrals**,

$$C^{-}_{\ell m\omega} := \int_{r_{\min}}^{+\infty} \frac{\psi^{+}_{\ell\omega}(r')S_{\ell m\omega}(r')}{W_{\ell\omega} f(r')} dr', \tag{25}$$

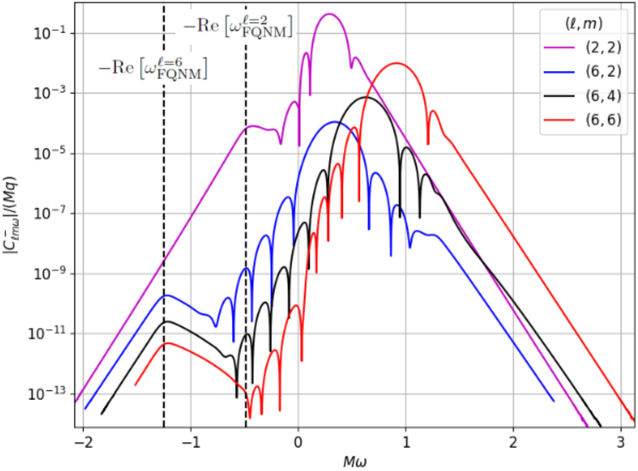
which stretch over the (unbounded) radial extent of the orbit.

- Slow, oscillatory convergence: problems when truncated at finite $r_{\max}$.
- Developed solutions:
  1. Tail corrections: use large-r approximation to integrand to derive analytical estimates to the neglected tail.
  2. Integration by parts (IBP): use IBP to increase decay rate of integrand.



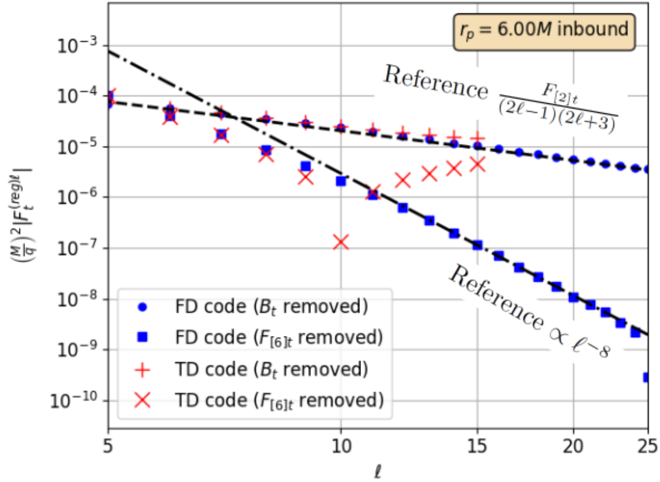
$C^-_{\ell m\omega}$ spectra

Example $C^-_{\ell m\omega}$ spectra for orbit $E = 1.1$, $r_{\min} = 4M$. Note QNM features.



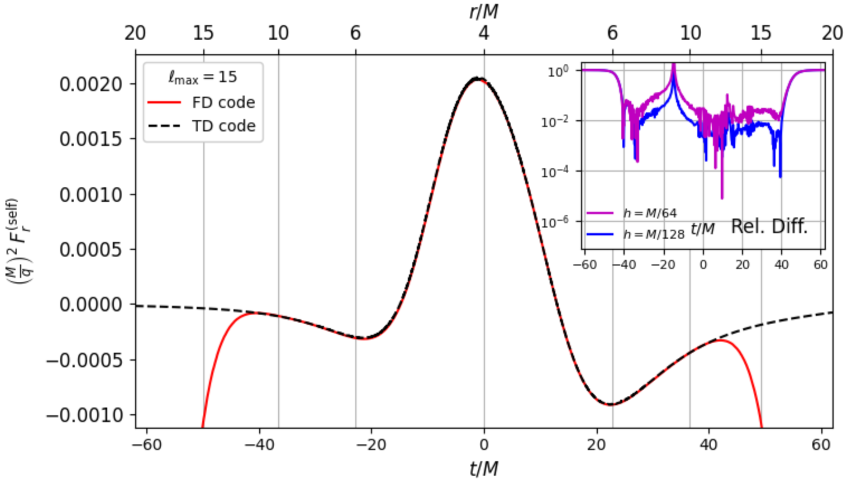
Self-force: regularisation tests

FD code agrees better with regularisation parameters at this radius



Self-force ($\ell_{\max} = 15$)

Good agreement with TD code near periapsis. Rapid deterioration in FD code as r increased.

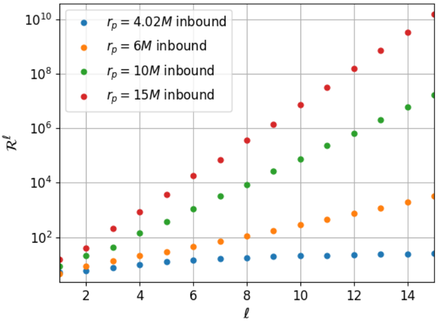


Cancellation problem

- Large-ℓ modes blow up rapidly with increasing radius.
- Low-frequency Fourier modes of the EHS field grow rapidly:

$$\tilde{\psi}^{-}_{\ell m\omega}(r) \sim r^{\ell+1} \quad (\omega r \ll 1). \tag{26}$$

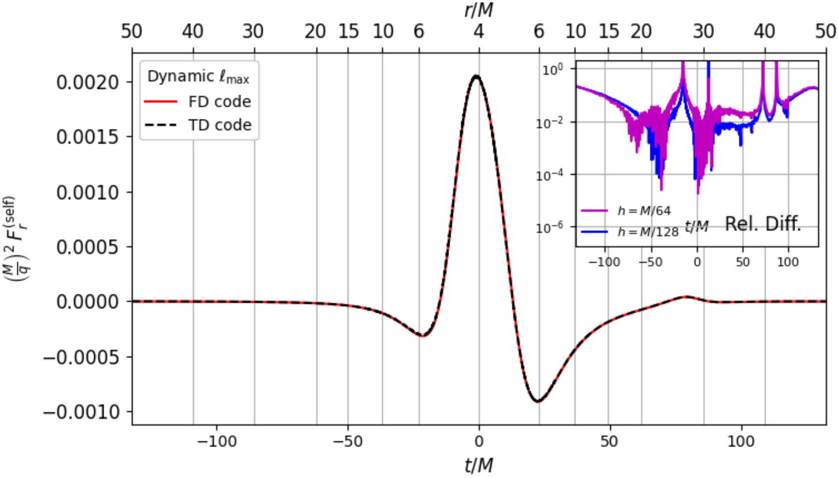
- Increasing cancellation between low-ω EHS modes to match physical TD field. [van de Meent 2016]
- Problem intrinsic to EHS method.



- Higher precision arithmetic unsuitable for scatter problem.
- We mitigate using dynamic ℓ truncation in the mode sum.

Self-force (dynamic $\ell_{\max}$)

Prevents catastrophic blow up, but still lose accuracy gradually.



PART D: Future work

Analytical calculation at large r (preliminary)

- Supplement FD code with analytic expansion of the SF in $1/r$.
- Makes use of a hierarchical expansion, [Barack 1998]

$$\psi_{\ell m}(u,v) = \sum_{N=0}^{\infty} \psi_N(u,v), \tag{27}$$

$$\psi_{0,uv} + V_0(r)\psi_0 = S(u,v), \tag{28}$$

$$\psi_{N,uv} + V_0(r)\psi_N = -\delta V(r)\psi_{N-1} \quad (N>0), \tag{29}$$

where $V_0(r)$ approximates asymptotic behaviour of exact potential $V(r)$, and $\delta V(r) := V(r) - V_0(r)$.

- ψ_0 (complete) does not contribute to SF; ψ_1 (underway) gives leading large-r behaviour.

PM resummation (preliminary)

- As $b \to b_c(v)$,

$$\chi_0 \sim A(v) \log\left(1 - \frac{b_c(v)}{b}\right) + \text{const}(v), \quad \delta\chi_{1\mathrm{SF}} \sim q_s B(v) \frac{b_c(v)}{b - b_c(v)}.$$

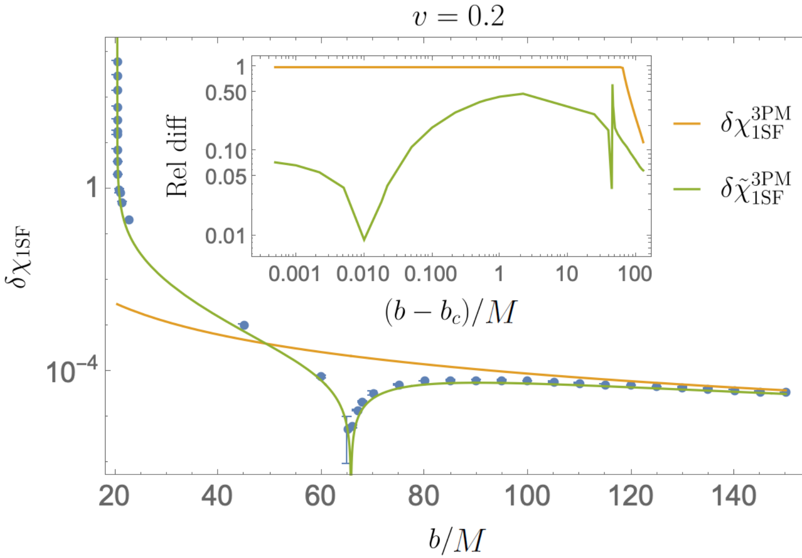
  $A(v)$ known analytically; $B(v)$ inferred from SF calculations.

- Consider the function

$$\Psi^{n\mathrm{PM}} = A\left[\log\left(1 - \frac{b_c(1 - q_s B/A)}{b}\right) + \sum_{k=1}^{n} \frac{1}{k}\left(\frac{b_c(1 - q_s B/A)}{b}\right)^k\right].$$

- We define the **resummed scatter angle** $\tilde{\chi}^{n\mathrm{PM}} := \chi^{n\mathrm{PM}} + \Psi^{n\mathrm{PM}}$.
  - Agrees with $\chi^{n\mathrm{PM}}$ through $n\mathrm{PM}$ order.
  - Matches the 0SF and 1SF divergences near separatrix.

PM resummation (preliminary)



Prospects

- Analytical results for SF at large r: useful for both TD and FD approaches.
- Improved TD methods also under development, including spectral methods with hyperboloidal slicing and compactification.
- Routes to gravity?
  - Direct Lorenz-gauge calculation [Ackay, Warburton, Barack 2013]
    - → Investigated extending h1Lorenz pacakge to unbound orbits with Warburton and Barack.
  - Radiation-gauge reconstruction [Pound, Merlin, Barack 2013]
  - Lorenz-gauge reconstruction [Dolan, Durkan, Kavanagh, Wardell 2023]
- Second order?
  - Easier than bound? No disparate timescales.
  - Would give conservative dynamics to 6PM.
  - Some way off.