Self-force in Hyperbolic Scattering: a Frequency Domain Approach

arXiv:2305.09724

Chris Whittall
with Leor Barack

26th Capra Meeting
4th July 2023

University of Southampton

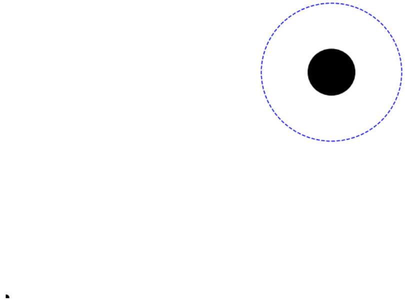
CAPRA

Scatter orbits

Particle starts at radial infinity at early times with velocity v and *impact parameter* b:

$$b = \lim_{\tau \to -\infty} r_p(\tau) \sin |\varphi_p(\tau) - \varphi_p(-\infty)| \,. \tag{1}$$

Provided $b > b_{\mathrm{crit}}(v)$, particle scatters off central black hole, approaching to within periapsis distance r_{min}.



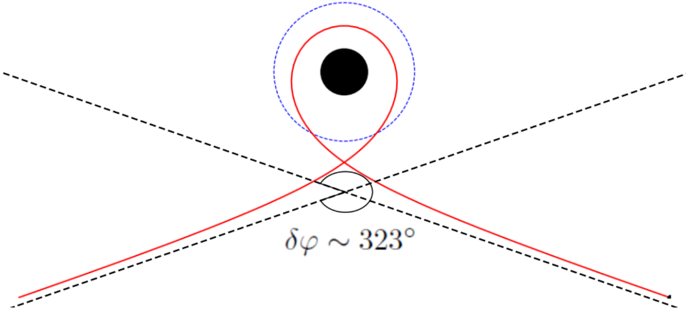
Scatter orbits

The *scatter angle* is defined:

$$\delta\varphi = \varphi_{\text{out}} - \varphi_{\text{in}} - \pi. \quad (2)$$

Fixing (b, v), this can be split into a geodesic part and SF corrections:

$$\delta\varphi = \delta\varphi^{(0)} + \eta\delta\varphi^{(1)} + \eta^2\delta\varphi^{(2)} + \ldots \quad (3)$$



Motivation

- Theoretical interest:
  - Clean, well-defined asymptotic in/out states
  - Probe strong-field (sub-ISCO) region even at low energies
- Boundary-to-bound [Kalin & Porto 2020] relations between scatter and bound orbit observables, derived using effective-field-theory.
- Conservative PM dynamics can be inferred from SF scatter angles, valid at *all* mass ratios [Damour 2020]:

$$1\text{SF} \implies 4\text{PM}.$$

- Comparison with quantum amplitude methods (e.g. double copy). [talks by Andres Luna, Olly Long]
- Benchmark PM results in the strong-field regime.
- PM results can be used to calibrate effective-one-body models.
  - Inform universal model of BBH inspirals, suitable for GW searches.
- Scatter orbits are unlikely observational candidates themselves.

Frequency-domain methods

- Time-domain (TD) methods provide a priori simplest route to self-force calculations along scatter orbits. [Barack and Long 2022]

- Frequency-domain (FD) methods valued for accuracy and efficiency with bound orbits.

- FD methods expected to retain these advantages, but challenges must be overcome:
  - Continuous spectrum.
  - Failure of EHS method.
  - Slowly convergent radial integrals.
  - Cancellation during TD reconstruction.

Use a scalar-field toy model in Schwarzschild to investigate and manage these problems. See arXiv:2305.09724 for details.

Scalar-field model

- The scalar field equation is given by

$$\nabla_\mu \nabla^\mu \Phi = -4\pi T, \tag{4}$$

where the scalar charge density T is that of a point particle.

- We separate into spherical and Fourier harmonics:

$$\Phi = \int d\omega \sum_{\ell,m} \frac{1}{r} \psi_{\ell m \omega}(r) Y_{\ell m}(\theta, \varphi) e^{-i\omega t}. \tag{5}$$

- The field equation becomes

$$\frac{d^2 \psi_{\ell m \omega}}{dr_*^2} - (V_\ell(r) - \omega^2)\psi_{\ell m \omega} = S_{\ell m \omega}(r). \tag{6}$$

Solutions

First step: introduce basis of homogeneous solutions $\psi^{\pm}_{\ell\omega}$ obeying retarded BCs at one boundary each:

$$\psi^{\pm}_{\ell\omega}(r) \to e^{\pm i\omega r_*} \qquad \text{as } r_* \to \pm\infty. \tag{7}$$

Two approaches:

1. Variation of parameters: solve for physical **inhomogeneous** field $\psi^{-}_{\ell m\omega}$.
    - SF reconstruction suffers from Gibbs phenomenon: slow ($\propto \omega^{-1}$), non-uniform convergence. Impractical.
2. Extended homogeneous solutions (EHS): reconstruct SF modes separately on either side of the orbit using suitably normalised frequency-domain **homogeneous** solutions.
    - Exponential, uniform convergence

Extended homogeneous solutions

- The EHS method relies crucially on the compactness of the source.
- EHS cannot a priori be used to reconstruct the SF modes in the "external" region $r \geq r_p(t)$ for unbounded orbits.
- For unbound orbits, SF modes in the "internal" region $r \leq r_p(t)$ may still be reconstructed from the frequency-domain EHS

$$\tilde{\psi}^{-}_{\ell m\omega}(r) := \psi^{-}_{\ell\omega}(r) \int_{r_{\min}}^{+\infty} \frac{\psi^{+}_{\ell\omega}(r') S_{\ell m\omega}(r')}{W_{\ell\omega} f(r')} dr'. \tag{8}$$

We use EHS and one-sided mode-sum regularisation

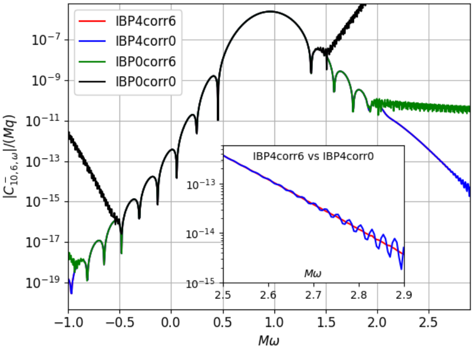
Truncation problem

- Need to evaluate the **normalisation integrals**,

$$C^{-}_{\ell m\omega} := \int_{r_{\min}}^{+\infty} \frac{\psi^{+}_{\ell\omega}(r')S_{\ell m\omega}(r')}{W_{\ell\omega}f(r')}dr', \tag{9}$$

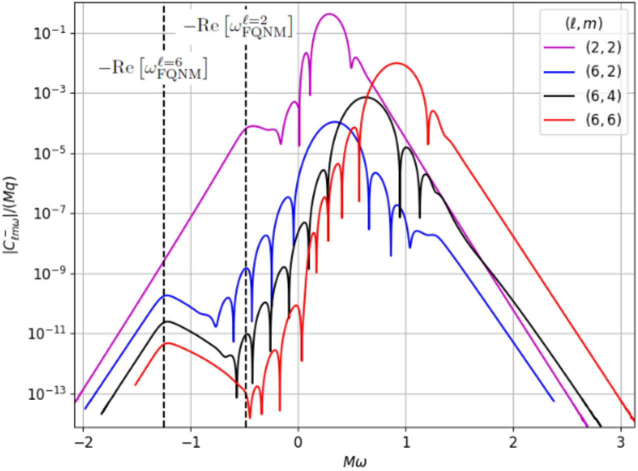
which stretch over the (unbounded) radial extent of the orbit.

- Slow, oscillatory convergence: problems when truncated at finite $r_{\max}$.
- Developed solutions:
  1. Tail corrections: use large-r approximation to integrand to derive analytical estimates to the neglected tail.
  2. Integration by parts (IBP): use IBP to increase decay rate of integrand.



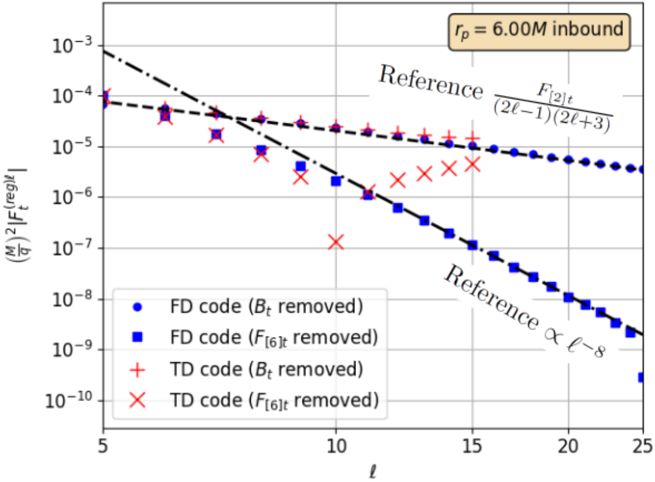
$C^-_{\ell m\omega}$ spectra

Example $C^-_{\ell m\omega}$ spectra for orbit $E = 1.1$, $r_{\min} = 4M$. Note QNM features.



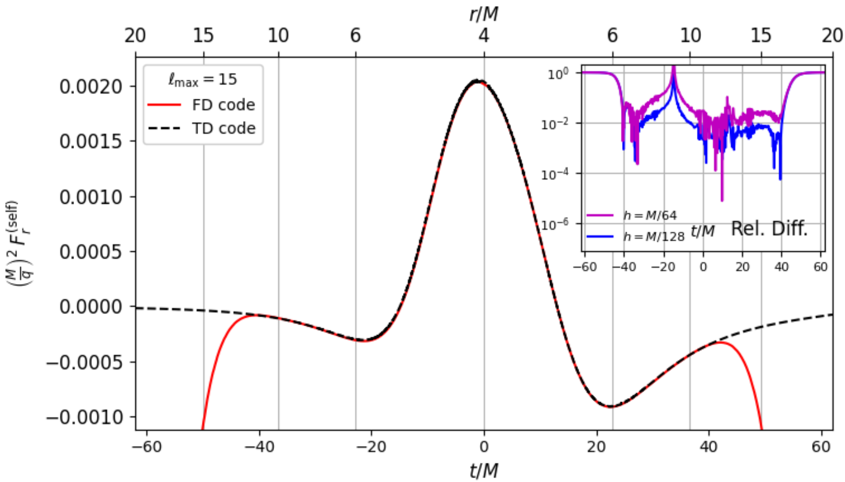
Self-force

FD code agrees better with regularisation parameters at this radius



Self-force

Good agreement with TD code near periapsis. Rapid deterioration in FD code as r increased.

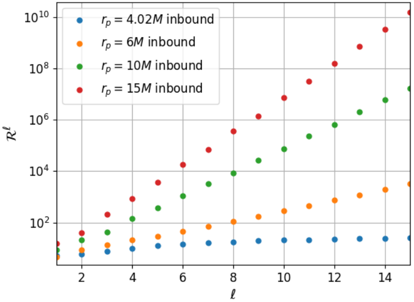


Cancellation problem

- Large-ℓ modes blow up rapidly with increasing radius.
- Low-frequency Fourier modes of the EHS field grow rapidly:

$$\tilde{\psi}^{-}_{\ell m\omega}(r) \sim r^{\ell+1} \quad (\omega r \ll 1). \tag{10}$$

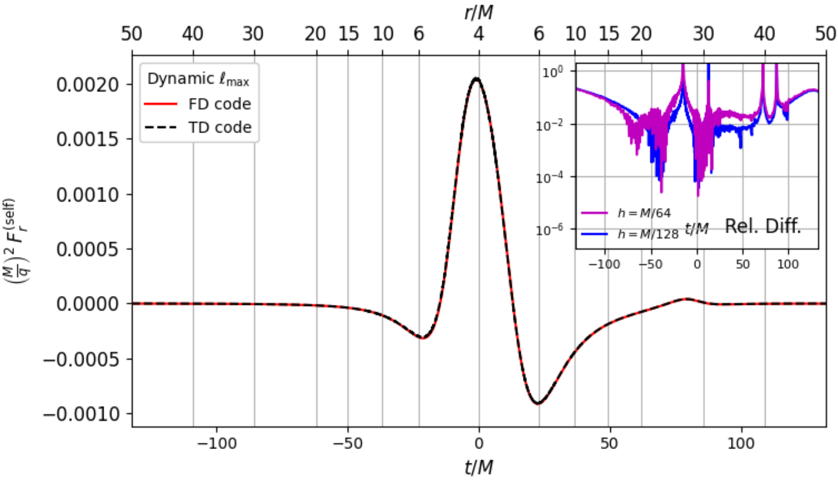
- Increasing cancellation between low-ω EHS modes to match physical TD field. [van de Meent 2016]
- Problem intrinsic to EHS method.



- Higher precision arithmetic unsuitable for scatter problem.
- We mitigate using dynamic ℓ truncation in the mode sum.

Self-force (dynamic $\ell_{\max}$)

Prevents catastrophic blow up, but still lose accuracy gradually.



Scatter angle

- Can use the FD-calculated SF to calculate scatter angle corrections:

$$\delta\varphi^{(1)}_{(\mathrm{cons})} \approx -1.5032, \quad \delta\varphi^{(1)}_{(\mathrm{diss})} \approx 2.7035 \quad (r_{\max} = 50M). \tag{11}$$

- Discrepancy of approx 1.8% (0.31%) in conservative (dissipative) piece compared to equivalent TD calculation.
- Compares to errors of approx 4.1% and 2.5% from truncating at $r_{\max} = 50M$.

Large-r issue is a limiting factor for the scatter angle calculation

Conclusion and outlook

We have demonstrated a frequency-domain method to calculate the self-force along hyperbolic geodesics in the Schwarzschild spacetime, overcoming several issues with the extension to unbound orbits:

- FD method displays superior accuracy to the TD code at smaller radii.
- FD method suffers rapidly loss of accuracy with increasing orbital radius due to known cancellation problem.

Future work (in collaboration with Leor Barack and Olly Long):

- Investigating benefits of FD/TD hybridisation.
- Investigating FD performance for weak-field orbits.
- Analytical calculation for self-force at early/late times.
- Investigate possible alternatives to the use of EHS.