Frequency Domain Approach to Self-Force in Hyperbolic Scattering

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Unbound Orbits in the Frequency Domain

- FD codes are faster than TD and more accurate for bound orbits.

- FD codes exist to calculate 1st order GSF on arbitrary bound Kerr geodesics [Van de Meent 2017].

- Using scalar field toy model (in Schwarzschild) to investigate and develop solutions to challenges with the unbound case:
  - Continuous spectrum
  - UV problem near particle and method of extended homogeneous solutions
  - Slowly convergent radial integral
  - IR problem

Equations of Motion

The scalar field equation of motion is given by

$$\nabla_\mu \nabla^\mu \Phi = -4\pi T \tag{1}$$

and the scalar charge density T is that of a point particle. We separate into spherical and Fourier harmonics:

$$\Phi = \int d\omega \sum_{\ell,m} \frac{1}{r} \psi_{\ell m\omega} Y_{\ell m}(\theta, \varphi) e^{-i\omega t}, \tag{2}$$

and the equation of motion becomes

$$\frac{d^2 \psi_{\ell m\omega}}{dr_*^2} - (V_\ell(r) - \omega^2)\psi_{\ell m\omega} = S_{\ell m\omega}(r). \tag{3}$$

Inhomogeneous Solutions (bound case)

Inhomogeneous solutions can be found using variation of parameters. Considering first a *bound* orbit:

$$\psi_{\ell m\omega}(r) = \psi^{+}_{\ell m\omega}(r) \int_{r_{min}}^{r} \frac{\psi^{-}_{\ell m\omega}(r') S_{\ell m\omega}(r')}{W_{\ell m\omega}} \frac{dr'}{f(r')} + \psi^{-}_{\ell m\omega}(r) \int_{r}^{r_{max}} \frac{\psi^{+}_{\ell m\omega}(r') S_{\ell m\omega}(r')}{W_{\ell m\omega}} \frac{dr'}{f(r')}$$

obeys the relevant BCs.

For $\omega \neq 0$ the homogeneous solutions $\psi^{\pm}_{\ell m\omega}$ are defined by BCs:

$$\psi^{-}_{\ell m\omega}(r) \sim e^{-i\omega r_*} \quad \text{as } r_* \longrightarrow -\infty \tag{4}$$

$$\psi^{+}_{\ell m\omega}(r) \sim e^{+i\omega r_*} \quad \text{as } r_* \longrightarrow +\infty. \tag{5}$$

Extended Homogeneous Solutions

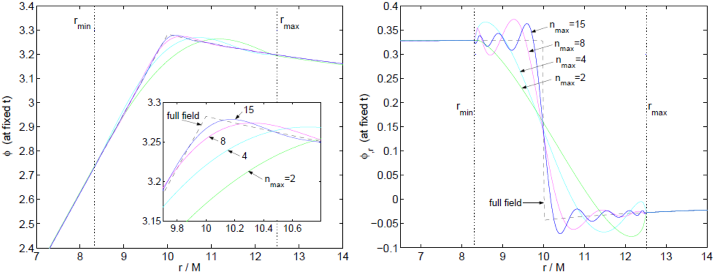


Figure: Convergence of Fourier series for scalar monopole (Schwarzschild eccentric orbit) [Barack, Ori, Sago 2008]

MEHS: express time domain field $\Phi_{lm}(t,r)$ in terms of analytic functions on either side of the worldline.

$$r\Phi_{\ell m}(t,r) = \tilde{\psi}^{+}_{\ell m}(t,r)\Theta(r - r_p(t)) + \tilde{\psi}^{-}_{\ell m}(t,r)\Theta(r_p(t) - r) \quad (6)$$

Extended Homogeneous Solutions (2)

First define the extended homogeneous solutions on $r > 2M$

$$\tilde{\psi}^{\pm}_{\ell m\omega}(r) := \psi^{\pm}_{\ell m\omega}(r) \int_{r_{min}}^{r_{max}} \frac{\psi^{\mp}_{\ell m\omega}(r') S_{\ell m\omega}(r') dr'}{W_{\ell m\omega} f(r')} \qquad (7)$$

and construct the corresponding time domain functions $\tilde{\psi}^{\pm}(t, r)$.

Key ideas:

1. In the source free region $r \geq r_{max}$, $\psi_{\ell m}(t, r) = \tilde{\psi}^{+}_{\ell m}(t, r)$.
2. $\psi_{\ell m}(t, r)$ and $\tilde{\psi}^{+}_{\ell m}(t, r)$ are analytic throughout $r > r_p(t)$.
3. Hence they must agree throughout $r \geq r_p(t)$.

Make a similar argument for $r \leq r_p(t)$.

[Barack, Ori, Sago 2008]

Inhomogeneous Solutions in the Unbound Case

For $\omega \neq 0$ variation of parameters again gives us the inhomogeneous field

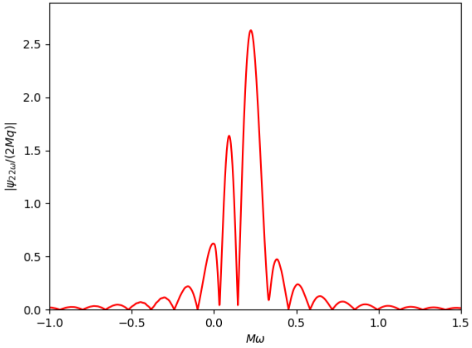
$$\begin{aligned} \psi_{\ell m\omega}(r) = {} & \psi^{+}_{\ell m\omega}(r) \int_{r_{min}}^{r} \frac{\psi^{-}_{\ell m\omega}(r') S_{\ell m\omega}(r')}{W_{\ell m\omega}} \frac{dr'}{f(r')} \\ & + \psi^{-}_{\ell m\omega}(r) \int_{r}^{\infty} \frac{\psi^{+}_{\ell m\omega}(r') S_{\ell m\omega}(r')}{W_{\ell m\omega}} \frac{dr'}{f(r')} \end{aligned} \tag{8}$$

Marginal convergence of this integral, integrand

$$\sim \frac{\text{oscillations}}{r}$$

as $r \longrightarrow \infty$.

Sample Spectrum



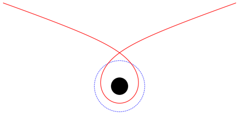
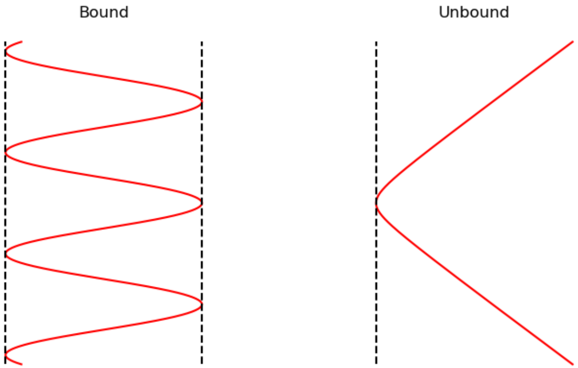


Figure: $\psi_{22\omega}(r = 6M)$ vs frequency for the geodesic $E = 1.1$, $r_{min} = 4M$ (illustrated).

EHS in unbound case: problem

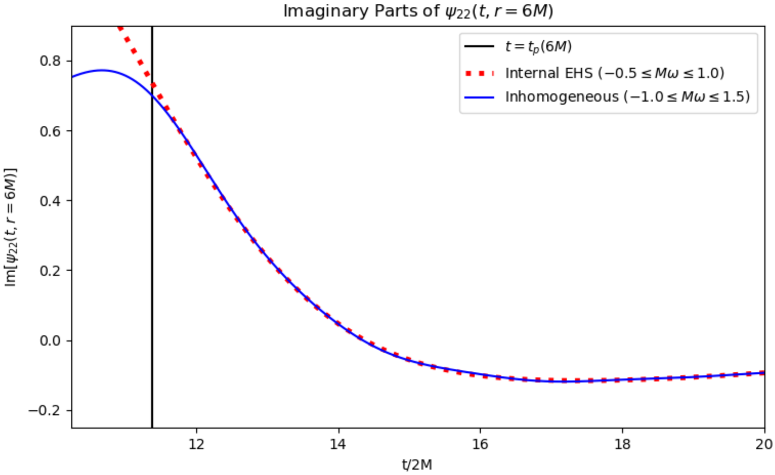
- Usual EHS argument fails in exterior region for unbound orbit: no source-free region $r \geq r_{max}$.
- Still holds for the interior region $r \leq r_p(t)$.



EHS in unbound case: possible solutions

- Could some form of EHS still apply?
- Extension into $u = 1/r < 0$:
  1. Scattering orbit extends to orbit in $u < 0$ region, periodic in Mino time.
  2. Need to find a global time coordinate which allows field equation to be separated into frequency modes before this is tractable.
- One-sided regularisation using only lower EHS.

Reconstructing Time Domain Field (Internal)



Slowly Converging Radial Integrals

- Need to compute radial integrals extending out to ∞, want to truncate at finite radius.

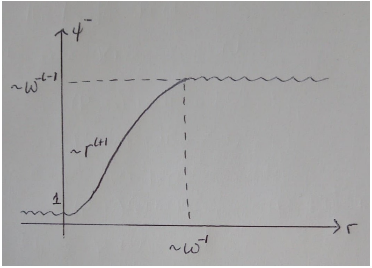
- Slow oscillations/r behaviour of integrand.

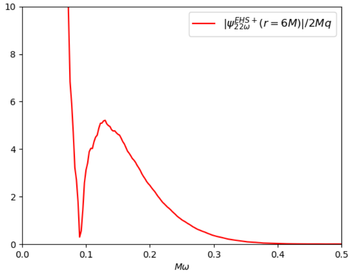
- In wave zone integrand can be expanded in $1/r$ and resulting integrals known analytically.
  1. Puncture integrand to get higher rate of convergence
  2. OR Analytical correction to truncated integral

- Particularly acute for external normalisation integral.

$$C^{+}_{\ell m\omega} = \int_{r_{min}}^{+\infty} \frac{\psi^{-}_{\ell m\omega}(r') S_{\ell m\omega}(r') dr'}{W_{\ell m\omega} f(r')} \tag{9}$$

The IR Problem

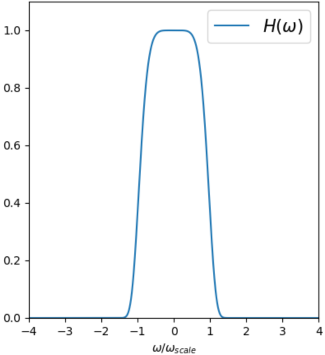




$$\psi^{EHS+}_{\ell m\omega}(r) = \psi^{+}_{\ell m\omega}(r) \int_{r_{min}}^{+\infty} \frac{\psi^{-}_{\ell m\omega}(r') S_{\ell m\omega}(r') dr'}{W_{\ell m\omega} f(r')} \tag{10}$$

Heuristics suggest genuine power law divergence...

Resolving the IR problem: Windowed EHS



Introduce a suitable window function, e.g.

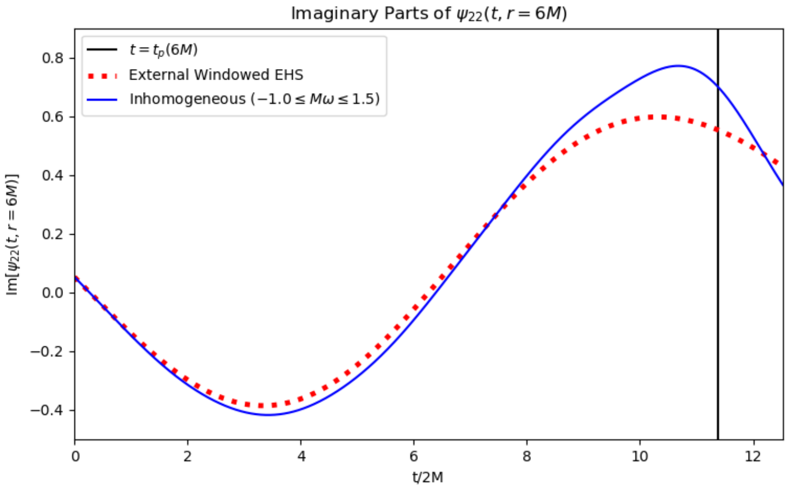
$$H(\omega) = \exp\left[-(\omega/\omega_{scale})^{2n}\right] \tag{11}$$

to split solution into high and low frequency parts.

Assuming EHS can be applied with usual form, outside the orbit:

$$\psi_{\ell m}(t,r) = \int_{-\infty}^{+\infty}\left[H(\omega)\psi^{inh}_{\ell m\omega}(r) + (1 - H(\omega))\psi^{EHS+}_{\ell m\omega}(r)\right]e^{-i\omega t}d\omega. \tag{12}$$

Reconstructing Time Domain Field (External)



Next Steps

- Work to understand how, if at all, we can apply EHS in external region.
- Dealing with numerical issues.
- Time domain reconstruction.
- SSF calculations and effect on scatter angle and time delay.
- Comparison with time-domain calculations (Oliver Long).